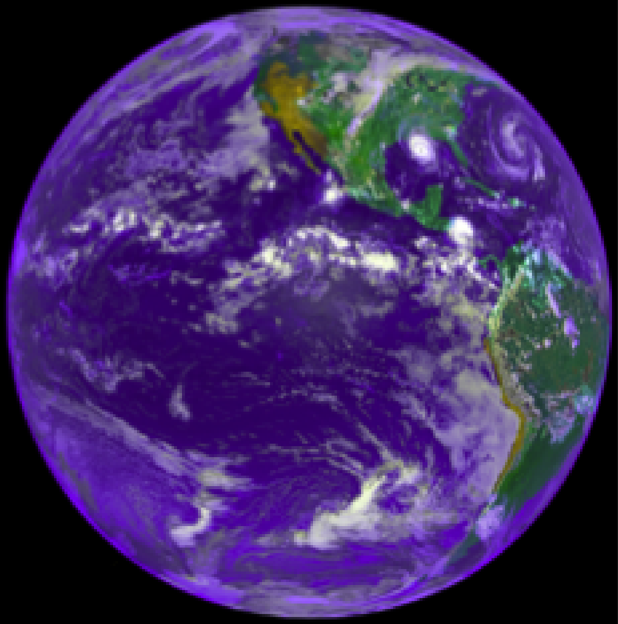
2013 Dolomites Research Week on Approximation



Lecture 2: Reconstruction and decomposition of vector fields on the sphere with applications

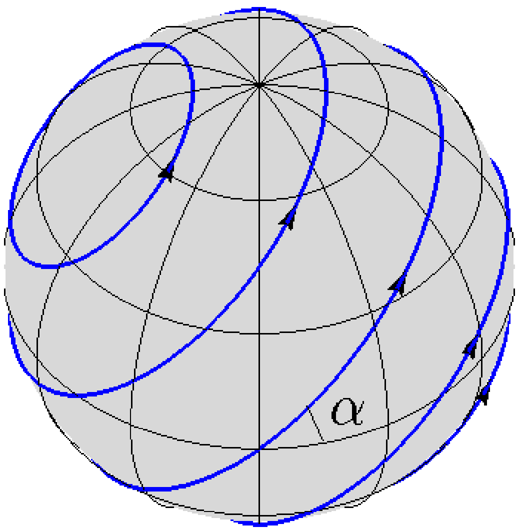
Grady B. Wright

Boise State University

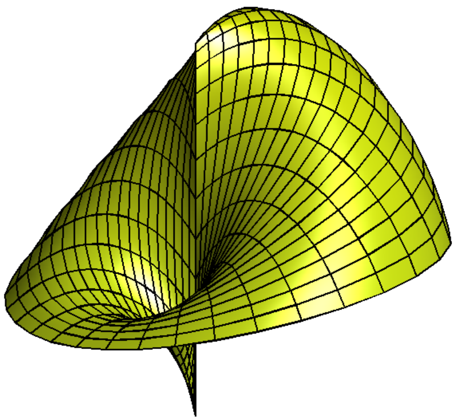
What's the problem with vector fields on the sphere?

- Consider tangent vector field for solid body rotation:

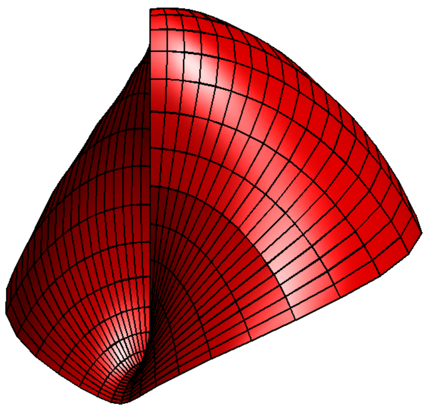
$$\mathbf{u} = [u_s, v_s]^T = u_0[(\cos\alpha\cos\theta + \sin\alpha\cos\lambda\sin\theta), -\sin\alpha\sin\lambda]^T$$



$r(\lambda,\theta) = 1 + u_s(\lambda,\theta)$



$r(\lambda,\theta) = 1 + v_s(\lambda,\theta)$



Components of $\mathbf{u}$ are discontinuous at poles!

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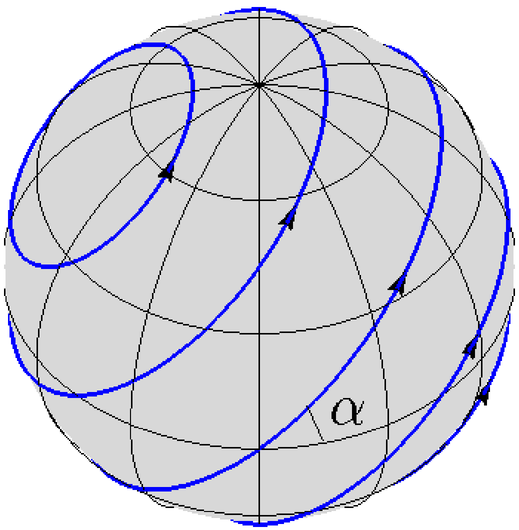
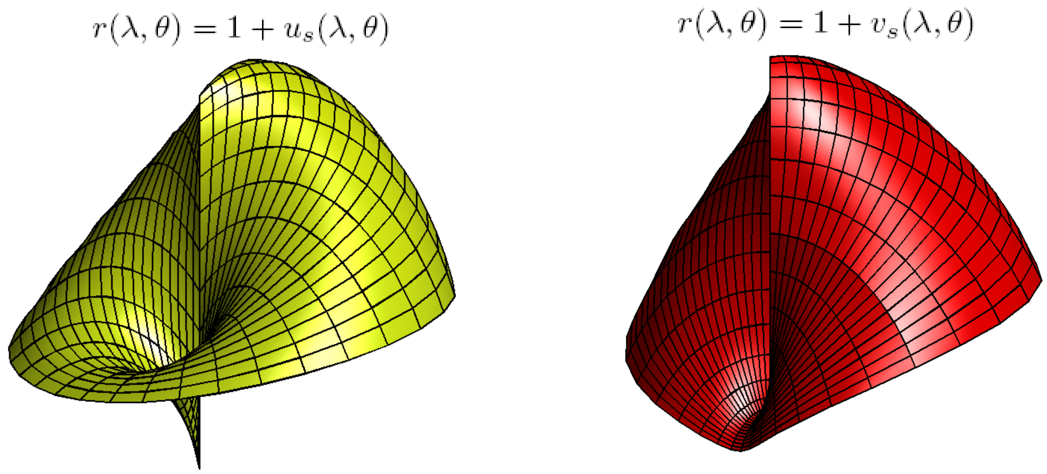




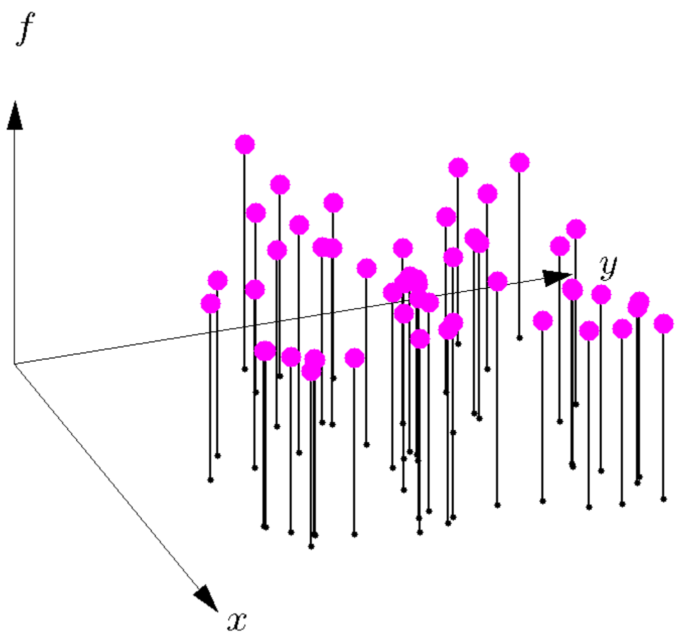
Components of $\mathbf{u}$ are discontinuous at poles!

- Purpose of this tutorial: Describe a reconstruction method for tangent vector fields that:

1. Is entirely free of any coordinate singularities.
2. Can accurately reconstruct a vector field from *scattered samples* of the field.
3. Analytically preserves certain physical properties of the field.
4. Can be differentiated to accurately compute the divergence and vorticity of the field.

- The construction is based on *positive definite matrix valued kernels*

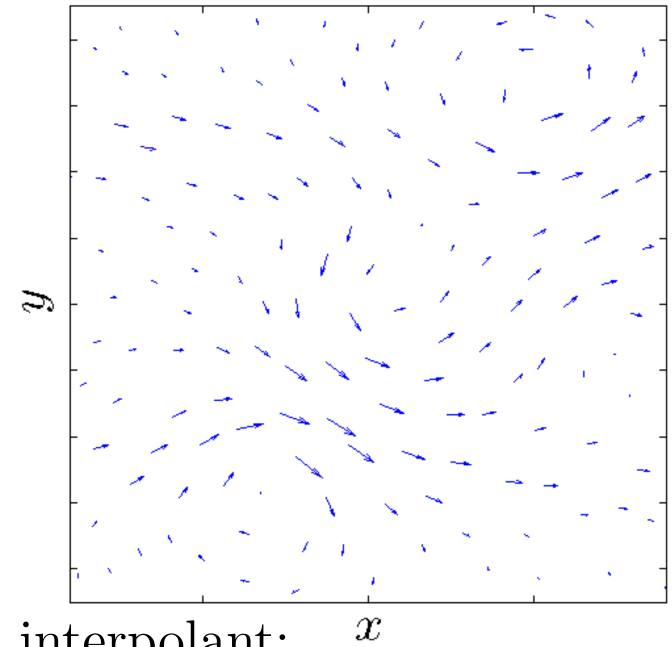
Scalar vs. vector interpolation with kernels



Scalar interpolant:

$$g(\mathbf{x}) = \sum_{j=1}^{N} \phi(\|\mathbf{x}-\mathbf{x}_j\|)b_j, \;\; g(\mathbf{x}_k) = f_k, \;\; k = 1, \ldots, N$$

- ϕ is scalar-valued "radial" kernel.
- Nodes can be "scattered".
- Interpolation matrix is positive definite.
- Can impose additional constraints on g.
- Form of the interpolant does not depend on the topology of domain.



Vector interpolant:

$$\mathbf{s}(\mathbf{x}) = \sum_{j=1}^{N} \Phi(\mathbf{x},\mathbf{x}_j)\mathbf{c}_j, \;\; \mathbf{s}(\mathbf{x}_k) = \mathbf{u}_k, \;\; k = 1, \ldots, N$$

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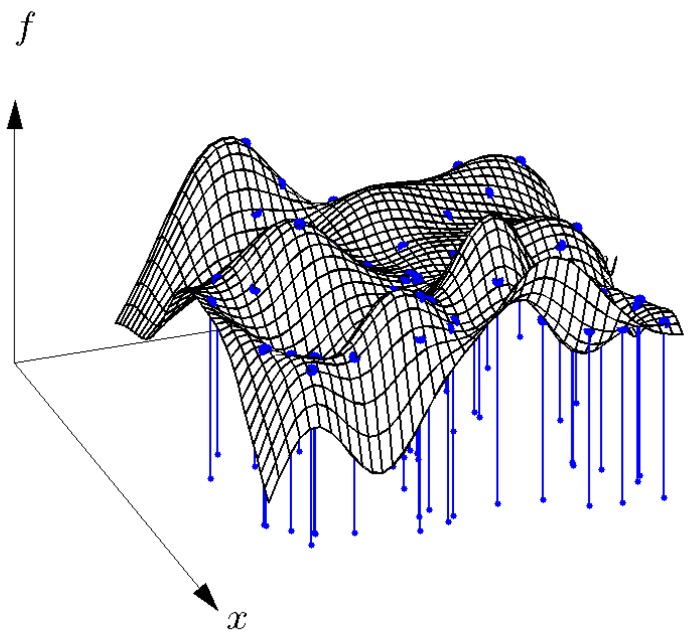
Scalar vs. vector interpolation with kernels



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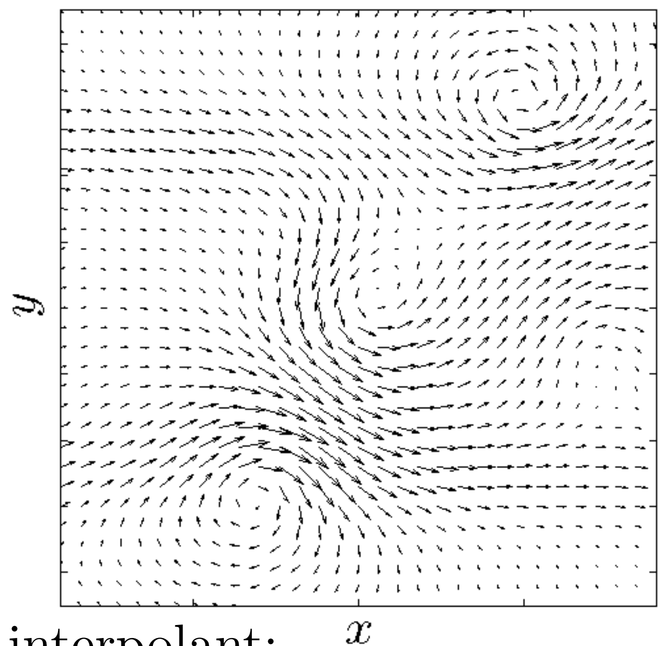

Vector interpolant:

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Background

- Customizing RBF approximations for vector fields:

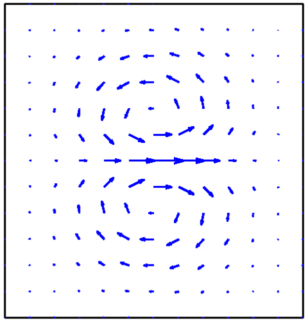
$$\mathbf{s}(\mathbf{x}) = \sum_{j=1}^{N} \Phi(\mathbf{x}, \mathbf{x}_j)\mathbf{c}_j, \;\; \mathbf{s}(\mathbf{x}_k) = \mathbf{u}_k, \;\; k = 1, \ldots, N$$

  - Developed by Narcowich and Ward (1994)

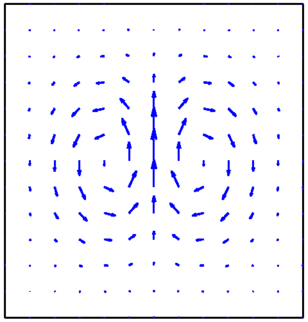
    F. J. Narcowich and J. D. Ward. Generalized Hermite interpolation via matrix-valued conditionally positive definite functions. *Math. Comp.*, 63:661-687, 1994.

  - Divergence-free vector fields: fluid flows, (static) magnetic fields

Columns of div-free matrix-valued kernel:



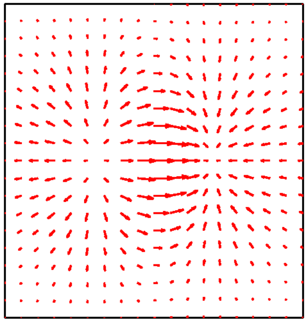
$\Phi_{\text{div}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



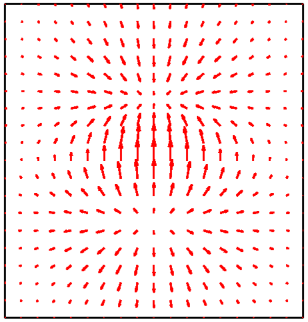
$\Phi_{\text{div}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

  - Curl-free vector fields: gravity fields, (static) electric fields

Columns of curl-free matrix-valued kernel:

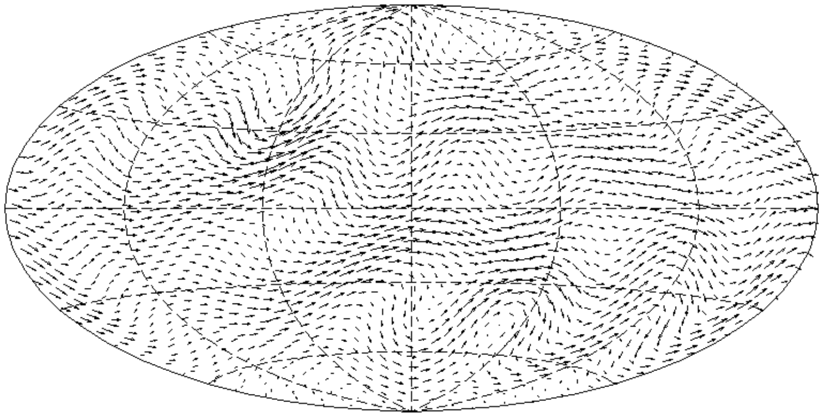


$\Phi_{\text{curl}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



$\Phi_{\text{curl}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Focus of this tutorial



- Kernel approximation of vector fields tangent to the surface of the sphere:
  - Surface divergence-free approximation
  - Surface curl-free approximation
  - Helmholtz-Hodge decomposition
    - F.J. Narcowich, J.D. Ward, and G.B.W. Divergence-free RBFs on Surfaces. *J. Fourier Anal. Appl.* 13 (2007), 643-663.
    - E.J. Fuselier, F.J. Narcowich, J.D. Ward, and G.B.W. Error and stability estimates for surface-divergence free RBF interpolants on the sphere. *Math. Comp.*, 78 (2009), 2157-2186.
    - E.J. Fuselier and G.B.W. Stability and error estimates for vector field interpolation and decomposition on the sphere with RBFs. *SIAM J. Numer. Anal.*, 47 (2009), 3213-3239.

- Geophysical applications

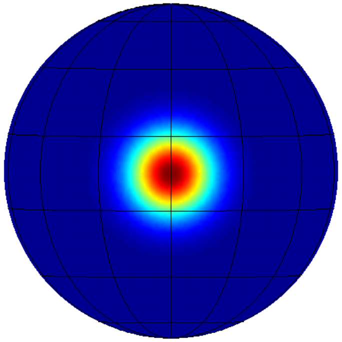
(Surface) Div, Grad, Curl, and all that

	Spherical Coords.	Cartesian Coords.
Point:	$(\lambda, \theta, 1)$	(x, y, z)
Unit vectors:	$\hat{\mathbf{i}} =$ longitudinal $\hat{\mathbf{j}} =$ latitudinal $\hat{\mathbf{k}} =$ radial	$\hat{\mathbf{i}} = x$-direction $\hat{\mathbf{j}} = y$-direction $\hat{\mathbf{k}} = z$-direction
Unit tangent vectors:	$\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$	$\boldsymbol{\zeta} = \frac{1}{\sqrt{1-z^2}}\begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}$, $\boldsymbol{\mu} = \frac{1}{\sqrt{1-z^2}}\begin{bmatrix} -zx \\ -zy \\ 1-z^2 \end{bmatrix}$
Unit normal vector:	$\hat{\mathbf{k}}$	$\mathbf{x} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$
Gradient of scalar g:	$\mathbf{u}_s = \nabla_s\, g = \frac{1}{\cos\theta}\frac{\partial g}{\partial \lambda}\hat{\mathbf{i}} + \frac{\partial g}{\partial \theta}\hat{\mathbf{j}}$	$\mathbf{u}_c = P_{\mathbf{x}}(\nabla_c\, g) = P_{\mathbf{x}}\left(\frac{\partial g}{\partial x}\hat{\mathbf{i}} + \frac{\partial g}{\partial y}\hat{\mathbf{j}} + \frac{\partial g}{\partial z}\hat{\mathbf{k}}\right)$
Surface divergence of $\mathbf{u}$:	$\nabla_s \cdot \mathbf{u}_s = \frac{1}{\cos\theta}\frac{\partial u_s}{\partial \lambda} + \frac{\partial v_s}{\partial \theta}$	$(P_{\mathbf{x}}\nabla_c)\cdot\mathbf{u}_c = \nabla_c\cdot\mathbf{u}_c - \mathbf{x}\cdot\nabla(\mathbf{u}_c\cdot\mathbf{x})$
Curl of a scalar f:	$\mathbf{u}_s = \hat{\mathbf{k}}\times(\nabla_s f) = -\frac{\partial f}{\partial \theta}\hat{\mathbf{i}} + \frac{1}{\cos\theta}\frac{\partial f}{\partial \lambda}\hat{\mathbf{j}}$	$\mathbf{u}_c = \mathbf{x}\times(P_{\mathbf{x}}\nabla_c f) = Q_{\mathbf{x}}P_{\mathbf{x}}(\nabla_c f) = Q_{\mathbf{x}}(\nabla_c f)$
Surface curl of a vector $\mathbf{u}$:	$\hat{\mathbf{k}}\cdot(\nabla_s\times\mathbf{u}_s) = -\nabla_s\cdot(\hat{\mathbf{k}}\times\mathbf{u}_s)$	$\mathbf{x}\cdot((P_{\mathbf{x}}\nabla_c)\times\mathbf{u}_c) = -\nabla_c\cdot(Q_{\mathbf{x}}\mathbf{u}_c)$

where
$$P_{\mathbf{x}} = I - \mathbf{x}\mathbf{x}^T = \begin{bmatrix} 1-x^2 & -xy & -xz \\ -xy & 1-y^2 & -yz \\ -xz & -yz & 1-z^2 \end{bmatrix} \quad \text{and} \quad Q_{\mathbf{x}} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

Surface divergence-free RBFs on the sphere

- Use extrinsic (Cartesian) coordinates, $\mathbf{x} = (x, y, z) \in \mathbb{S}^2$.
- Start with a radial kernel centered $\mathbf{y} \in \mathbb{S}^2$: $\phi(\|\mathbf{x} - \mathbf{y}\|)$



- Construct 3-by-3 *matrix-valued* function

$$\begin{aligned}\Psi_{\text{div}}(\mathbf{x}, \mathbf{y}) &= (Q_{\mathbf{x}}\nabla_{\mathbf{x}})(Q_{\mathbf{y}}\nabla_{\mathbf{y}})^T \phi(\|\mathbf{x} - \mathbf{y}\|) \\ &= Q_{\mathbf{x}}(\nabla\nabla^T \phi(\|\mathbf{x} - \mathbf{y}\|))Q_{\mathbf{y}}\end{aligned}$$

- If $\mathbf{c} = (c_1, c_2, c_3)^T$ is tangent to $\mathbb{S}^2$ at $\mathbf{y}$ then $\Psi_{\text{div}}(\mathbf{x}, \mathbf{y})\mathbf{c}$ is tangent to $\mathbb{S}^2$ at $\mathbf{x}$.

- Furthermore,

$$\begin{aligned}\Psi_{\text{div}}(\mathbf{x}, \mathbf{y})\mathbf{c} &= \left[Q_{\mathbf{x}}(\nabla\nabla^T \phi(\|\mathbf{x} - \mathbf{y}\|))Q_{\mathbf{y}}\right]\mathbf{c} \\ &= Q_{\mathbf{x}}\nabla\left[\nabla^T\left(\phi(\|\mathbf{x} - \mathbf{y}\|)Q_{\mathbf{y}}\mathbf{c}\right)\right] \\ &= Q_{\mathbf{x}}(\nabla f).\end{aligned}$$

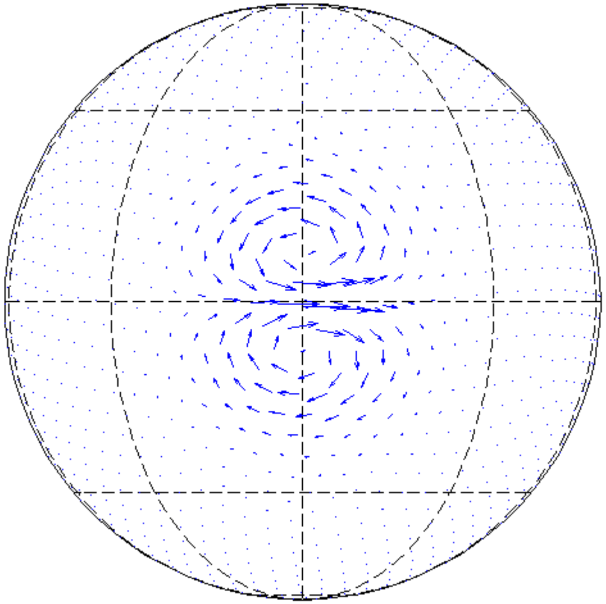
  Thus, $\Psi_{\text{div}}(\mathbf{x}, \mathbf{y})\mathbf{c}$ is surface divergence-free.

- Idea can be extended to other smooth orientable manifolds.

Surface div-free RBF interpolation

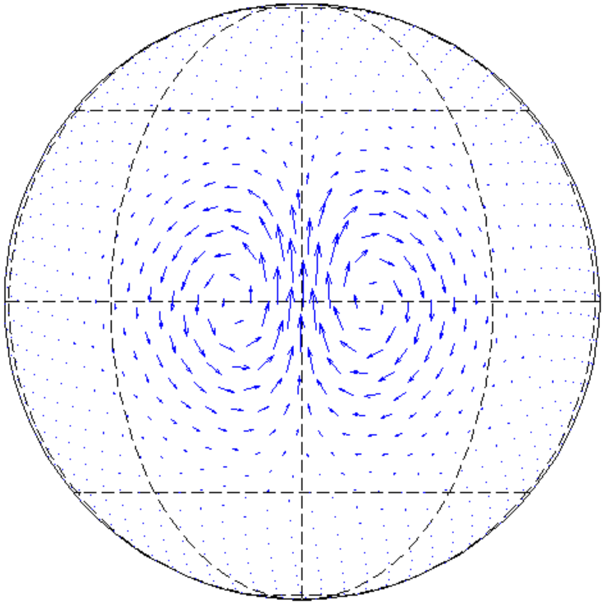
- Illustration of new basis (orthographic projection):

Zonal basis



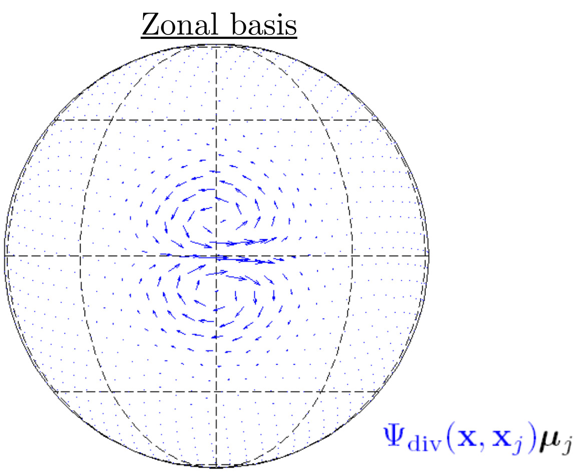
$\Psi_{\text{div}}(\mathbf{x}, \mathbf{x}_j)\boldsymbol{\mu}_j$

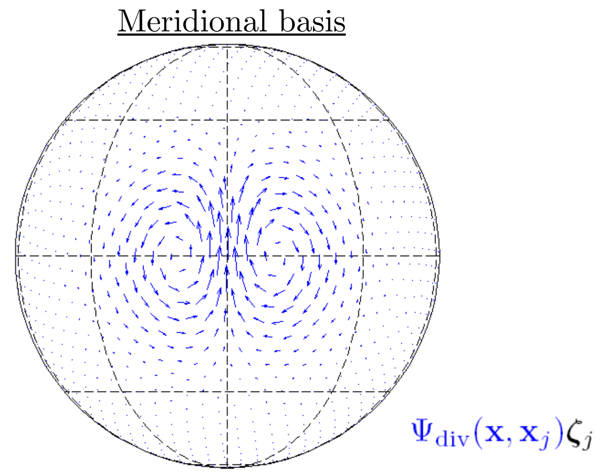
Meridional basis



$\Psi_{\text{div}}(\mathbf{x}, \mathbf{x}_j)\boldsymbol{\zeta}_j$

Surface div-free RBF interpolation





Procedure:

1. For each distinct node $\mathbf{x}_j$, center a $\Psi_{\text{div}}(\mathbf{x},\mathbf{x}_j)\boldsymbol{\mu}_j$ and $\Psi_{\text{div}}(\mathbf{x},\mathbf{x}_j)\boldsymbol{\zeta}_j$.
2. Linearly combine these *vector-valued* functions to satisfy the interpolation conditions.

$$\mathbf{s}(\mathbf{x}) = \sum_{j=1}^{N} \Psi_{\text{div}}(\mathbf{x},\mathbf{x}_j) \underbrace{\left[a_j\boldsymbol{\mu}_j + b_j\boldsymbol{\zeta}_j\right]}_{\mathbf{c}_j}$$

3. Solve $2N$-by-$2N$ linear system for unknown coefficients.

Interpolant will

- exist and be unique (linear system positive definite),
- be tangent to the sphere,
- be free of any pole singularity,
- be surface divergence-free.

Existence and uniqueness of the div-free interpolant

- Strategy: Show the $\Psi_{\text{div}}(\mathbf{x},\mathbf{y})$ is positive definte on $\mathbb{S}^2$, i.e.:

$$\sum_{j,k} \mathbf{c}_k^T \Psi_{\text{div}}(\mathbf{x}_k,\mathbf{x}_j)\mathbf{c}_j > 0$$

for all distinct finite point sets $X=\{\mathbf{x}_j\}_{j=1}^N \subset \mathbb{S}^2$ and all $\mathbf{c}_j$ tangent to $\mathbb{S}^2$ at $\mathbf{x}_j$.

- Key: All positive definite radial kernels are zonal on $\mathbb{S}^2$ with the spectral representation:

$$\phi(\|\mathbf{x}-\mathbf{y}\|) = \psi(\mathbf{x}\cdot\mathbf{y}) = \sum_{\ell=0}^{\infty} \widehat{\psi}(\ell) \sum_{m=-\ell}^{\ell} Y_\ell^m(\mathbf{x}) Y_\ell^m(\mathbf{y}) \qquad (\text{with } \widehat{\psi}(\ell) > 0)$$

- Thus we can write $\Psi_{\text{div}}(\mathbf{x},\mathbf{y}) := (Q_\mathbf{x}\nabla_\mathbf{x})(Q_\mathbf{y}\nabla_\mathbf{y})^T\phi(\|\mathbf{x}-\mathbf{y}\|)$ as

$$\begin{aligned}
\Psi_{\text{div}}(\mathbf{x},\mathbf{y}) &= (Q_\mathbf{x}\nabla_\mathbf{x})(Q_\mathbf{y}\nabla_\mathbf{y})^T\left(\sum_{\ell=0}^{\infty}\widehat{\psi}(\ell)\sum_{m=-\ell}^{\ell} Y_\ell^m(\mathbf{x})Y_\ell^m(\mathbf{y})\right) \\
&= \sum_{\ell=1}^{\infty}\widehat{\psi}(\ell)\sum_{m=-\ell}^{\ell}[(Q_\mathbf{x}\nabla_\mathbf{x})Y_\ell^m(\mathbf{x})][(Q_\mathbf{y}\nabla_\mathbf{y})^TY_\ell^m(\mathbf{y})] = \sum_{\ell=1}^{\infty}\ell(\ell+1)\widehat{\psi}(\ell)\sum_{m=-\ell}^{\ell}\mathbf{Y}_\ell^m(\mathbf{x})[\mathbf{Y}_\ell^m(\mathbf{y})]^T
\end{aligned}$$

- Using this result we immediately obtain:

$$\begin{aligned}
\sum_{j,k=1}^{N}\mathbf{c}_k^T\Psi_{\text{div}}(\mathbf{x}_k,\mathbf{x}_j)\mathbf{c}_j &= \sum_{j,k}\sum_{\ell=1}^{\infty}\ell(\ell+1)\widehat{\psi}(\ell)\sum_{m=-\ell}^{\ell}\mathbf{c}_k^T\mathbf{Y}_\ell^m(\mathbf{x}_k)[\mathbf{Y}_\ell^m(\mathbf{x}_j)]^T\mathbf{c}_j \\
&= \sum_{\ell=1}^{\infty}\ell(\ell+1)\widehat{\psi}(\ell)\sum_{m=-\ell}^{\ell}\left|\sum_{k=1}^{N}\mathbf{c}_k^T\mathbf{Y}_\ell^m(\mathbf{x}_k)\right|^2 \geq 0.
\end{aligned}$$

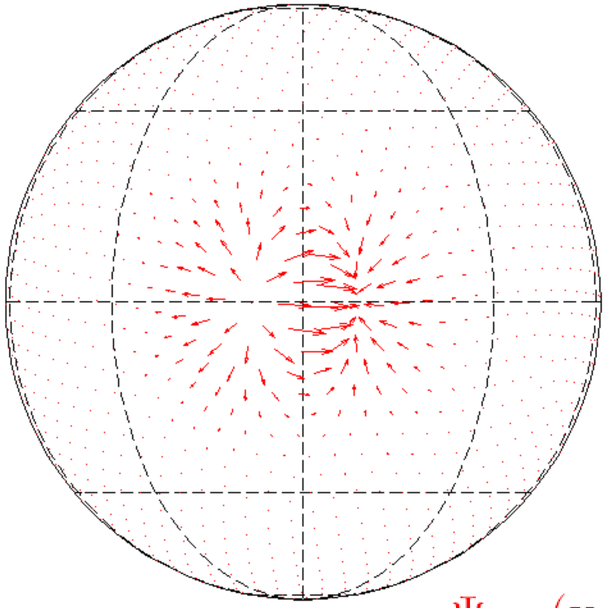
Surface curl-free RBF interpolation

- Surface curl-free basis:
  - Use extrinsic (Cartesian) coordinates, $\mathbf{x} = (x, y, z) \in \mathbb{S}^2$.
  - $\Psi_{\text{curl}}(\mathbf{x}, \mathbf{y}) = (P_{\mathbf{x}}\nabla_{\mathbf{x}})(P_{\mathbf{y}}\nabla_{\mathbf{y}}^T)\phi(\|\mathbf{x} - \mathbf{y}\|) = -P_{\mathbf{x}}(\nabla\nabla^T\phi(\|\mathbf{x} - \mathbf{y}\|))P_{\mathbf{y}}$
  - If $\mathbf{c} = (c_1, c_2, c_3)^T$ is tangent to $\mathbb{S}^2$ at $\mathbf{y}$ then $\Psi_{\text{curl}}(\mathbf{x}, \mathbf{y})\mathbf{c}$ is tangent to $\mathbb{S}^2$ at $\mathbf{x}$.
  - Furthermore, $\Psi_{\text{curl}}(\mathbf{x}, \mathbf{y})\mathbf{c} = P_{\mathbf{x}}\nabla \underbrace{\left[-\nabla^T\phi(\|\mathbf{x} - \mathbf{y}\|))P_{\mathbf{y}}\mathbf{c}\right]}_{g} = P_{\mathbf{x}}(\nabla g)$.

    Thus, $\Psi_{\text{curl}}(\mathbf{x}, \mathbf{y})\mathbf{c}$ is surface curl-free.

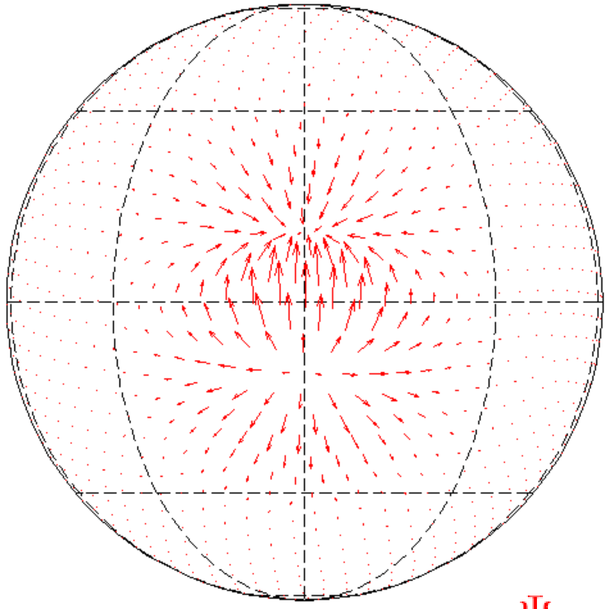
- Illustration of new basis (orthographic projection):

Zonal basis



$\Psi_{\text{curl}}(\mathbf{x}, \mathbf{x}_j)\boldsymbol{\mu}_j$

Meridional basis



$\Psi_{\text{curl}}(\mathbf{x}, \mathbf{x}_j)\boldsymbol{\zeta}_j$

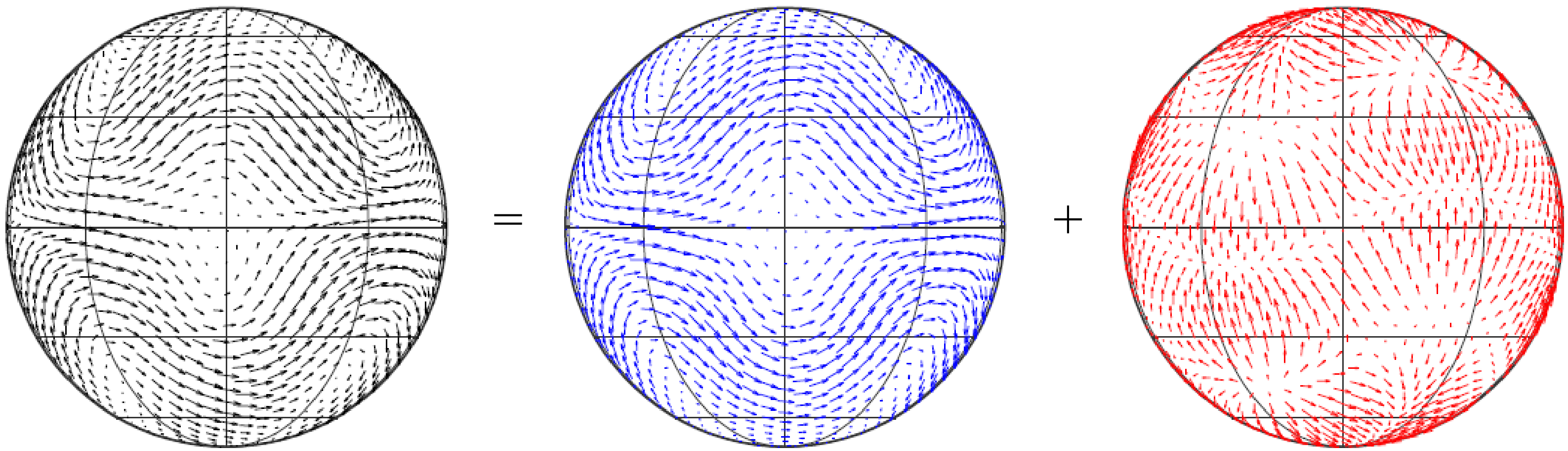
Helmholtz-Hodge Theorem

- <ins>Theorem:</ins> Any vector field tangent to the sphere can be *uniquely* decomposed into surface divergence-free and surface curl-free components:

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \mathbf{u}_{\text{div}}(\mathbf{x}) + \mathbf{u}_{\text{curl}}(\mathbf{x}) \\ &= Q_{\mathbf{x}}\nabla\psi(\mathbf{x}) + P_{\mathbf{x}}\nabla\chi(\mathbf{x}) \end{aligned}$$

ψ = stream function and χ = velocity potential

- Example:



- Goal: Construct an interpolant that mimics the Helmholtz-Hodge decomposition.

Helmholtz-Hodge Interpolant/Approximant

- Theorem: Any vector field tangent to the sphere can be *uniquely* decomposed into surface divergence-free and surface curl-free components:

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \mathbf{u}_{\text{div}}(\mathbf{x}) + \mathbf{u}_{\text{curl}}(\mathbf{x}) \\ &= Q_{\mathbf{x}}\nabla\psi(\mathbf{x}) + P_{\mathbf{x}}\nabla\chi(\mathbf{x}) \end{aligned}$$

ψ = stream function and χ = velocity potential

- RBF interpolant mimicking the Helmholtz-Hodge decomposition: $\mathbf{x} = (x, y, z) \in \mathbb{S}^2$

$$\begin{aligned} \mathbf{s}(\mathbf{x}) &= \sum_{j=1}^{N} \Psi(\mathbf{x}, \mathbf{x}_j)\mathbf{c}_j \qquad \text{(where } \mathbf{s}(\mathbf{x}_j) = \mathbf{u}_j,\ j = 1, \ldots, N\text{)} \\ &= \sum_{j=1}^{N} \left[\Psi_{\text{div}}(\mathbf{x}, \mathbf{x}_j) + \Psi_{\text{curl}}(\mathbf{x}, \mathbf{x}_j)\right]\mathbf{c}_j \\ &= \underbrace{\sum_{j=1}^{N} \Psi_{\text{div}}(\mathbf{x}, \mathbf{x}_j)\mathbf{c}_j}_{\approx\, \mathbf{u}_{\text{div}}} + \underbrace{\sum_{j=1}^{N} \Psi_{\text{curl}}(\mathbf{x}, \mathbf{x}_j)\mathbf{c}_j}_{\approx\, \mathbf{u}_{\text{curl}}} \end{aligned}$$

- Can get an approximation to the surface divergence-free and surface curl-free components!

Helmholtz-Hodge Interpolant/Approximant

- Theorem: Any vector field tangent to the sphere can be *uniquely* decomposed into surface divergence-free and surface curl-free components:

$$\begin{aligned}\mathbf{u}(\mathbf{x}) &= \mathbf{u}_{\text{div}}(\mathbf{x}) + \mathbf{u}_{\text{curl}}(\mathbf{x}) \\ &= Q_{\mathbf{x}}\nabla\psi(\mathbf{x}) + P_{\mathbf{x}}\nabla\chi(\mathbf{x})\end{aligned}$$

ψ = stream function and χ = velocity potential

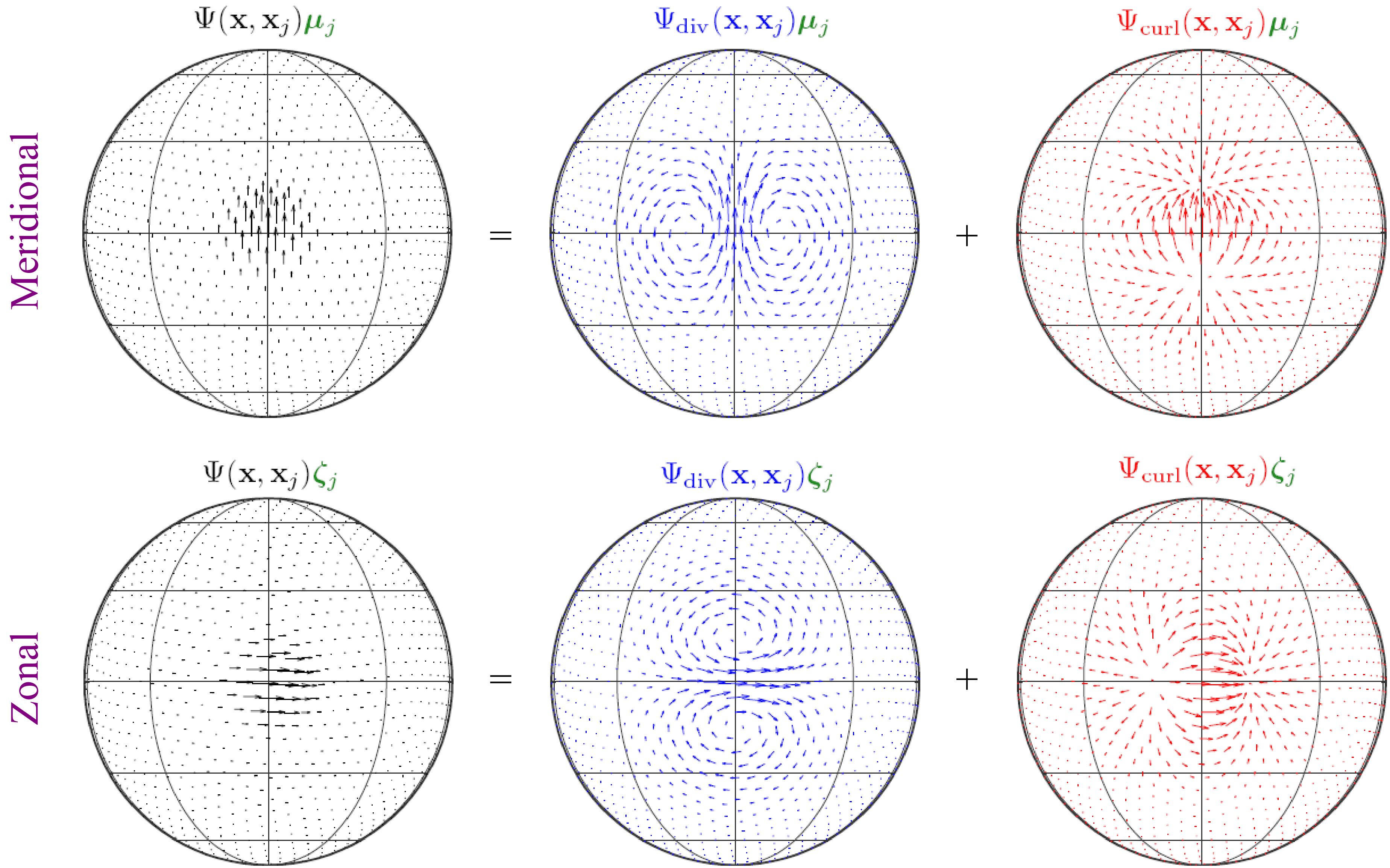
- RBF interpolant mimicking the Helmholtz-Hodge decomposition: $\mathbf{x} = (x, y, z) \in \mathbb{S}^2$

$$\begin{aligned}\mathbf{s}(\mathbf{x}) &= \sum_{j=1}^{N} \Psi(\mathbf{x}, \mathbf{x}_j)\mathbf{c}_j \qquad (\text{where } \mathbf{s}(\mathbf{x}_j) = \mathbf{u}_j,\ j = 1, \ldots, N) \\ &= \sum_{j=1}^{N} \left[\Psi_{\text{div}}(\mathbf{x}, \mathbf{x}_j) + \Psi_{\text{curl}}(\mathbf{x}, \mathbf{x}_j)\right]\mathbf{c}_j \\ &= Q_{\mathbf{x}}\nabla \underbrace{\left[\sum_{j=1}^{N} \nabla^T \phi(\|\mathbf{x} - \mathbf{x}_j\|) Q_{\mathbf{x}_j}^T \mathbf{c}_j\right]}_{\text{stream function for } \mathbf{s}} + P_{\mathbf{x}}\nabla \underbrace{\left[\sum_{j=1}^{N} \nabla^T \phi(\|\mathbf{x} - \mathbf{x}_j\|) P_{\mathbf{x}_j}^T \mathbf{c}_j\right]}_{\text{velocity potential for } \mathbf{s}}\end{aligned}$$

- Can get a stream function and velocity potential for the interpolant!

Illustration of new Helmholtz-Hodge RBF

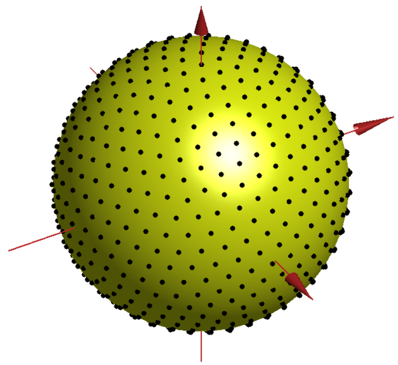
Interpolant: $\mathbf{s}(\mathbf{x}) = \sum_{j=1}^{N} \Psi(\mathbf{x},\mathbf{x}_j)\mathbf{c}_j = \sum_{j=1}^{N} \Psi_{\text{div}}(\mathbf{x},\mathbf{x}_j)\mathbf{c}_j + \sum_{j=1}^{N} \Psi_{\text{curl}}(\mathbf{x},\mathbf{x}_j)\mathbf{c}_j$, where $\mathbf{c}_j = a_j\boldsymbol{\mu}_j + b_j\boldsymbol{\zeta}_j$



Overview of Helmholtz-Hodge RBF properties

Interpolant: $\mathbf{s}(\mathbf{x}) = \sum_{j=1}^{N} \Psi(\mathbf{x},\mathbf{x}_j)\mathbf{c}_j = \sum_{j=1}^{N} \Psi_{\mathrm{div}}(\mathbf{x},\mathbf{x}_j)\mathbf{c}_j + \sum_{j=1}^{N} \Psi_{\mathrm{curl}}(\mathbf{x},\mathbf{x}_j)\mathbf{c}_j$, where $\mathbf{c}_j = a_j\boldsymbol{\mu}_j + b_j\boldsymbol{\zeta}_j$

- Construction of the interpolant identical to the div-free construction.
- Can show for any distinct node set, existence and uniqueness are guaranteed.
- Error estimates for reconstructing and decomposing vector field $\mathbf{u} = \mathbf{u}_{\mathrm{div}} + \mathbf{u}_{\mathrm{curl}}$: Key quantities:



$$\text{Separation radius: } q_X := \frac{1}{2}\min_{i\neq j} d(\mathbf{x}_i,\mathbf{x}_j)$$

$$\text{Mesh norm: } h_X := \sup_{\mathbf{x}\in\mathbb{S}^2}\min_{\mathbf{x}_j\in X} d(\mathbf{x},\mathbf{x}_j)$$

$$\text{Mesh ratio: } \rho_X = h_X/q_X$$

$$\text{Decay Fourier transform } \phi : \widehat{\phi} \sim (1+\|\boldsymbol{\xi}\|_2^2)^{-(\tau+\frac{3}{2})}$$

Theorem 1 (Fuselier and Wright 2009):
If $\mathbf{u}\in H^{\beta}(\mathbb{S}^2)$, with $\beta \geq \tau > 1$ then we have

$$\begin{aligned}
\|\mathbf{u}-\mathbf{s}\|_{L^2(\mathbb{S}^2)} &\leq C h_X^{\tau}\|\mathbf{u}\|_{H^{\tau}(\mathbb{S}^2)},\\
\|\mathbf{u}_{\mathrm{div}}-\mathbf{s}_{\mathrm{div}}\|_{L^2(\mathbb{S}^2)} &\leq C h_X^{\tau}\|\mathbf{u}\|_{H^{\tau}(\mathbb{S}^2)},\\
\|\mathbf{u}_{\mathrm{curl}}-\mathbf{s}_{\mathrm{curl}}\|_{L^2(\mathbb{S}^2)} &\leq C h_X^{\tau}\|\mathbf{u}\|_{H^{\tau}(\mathbb{S}^2)}.
\end{aligned}$$

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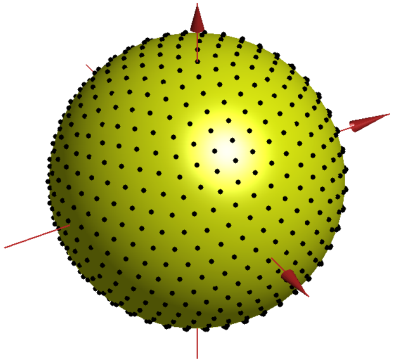

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Theorem 2 (Fuselier and Wright 2009):
If $\mathbf{u}\in H^{2\tau}(\mathbb{S}^2)$, with $\tau > 1$ then we have

$$\|\mathbf{u}-\mathbf{s}\|_{L^2(\mathbb{S}^2)} \le Ch_X^{2\tau}\|\mathbf{u}\|_{H^{2\tau}(\mathbb{S}^2)},$$

$$\|\mathbf{u}_{\text{div}}-\mathbf{s}_{\text{div}}\|_{L^2(\mathbb{S}^2)} \le Ch_X^{2\tau}\|\mathbf{u}\|_{H^{2\tau}(\mathbb{S}^2)},$$

$$\|\mathbf{u}_{\text{curl}}-\mathbf{s}_{\text{curl}}\|_{L^2(\mathbb{S}^2)} \le Ch_X^{2\tau}\|\mathbf{u}\|_{H^{2\tau}(\mathbb{S}^2)}.$$

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Interpolant: $\mathbf{s}(\mathbf{x}) = \sum_{j=1}^{N} \Psi(\mathbf{x}, \mathbf{x}_j)\mathbf{c}_j = \sum_{j=1}^{N} \Psi_{\text{div}}(\mathbf{x}, \mathbf{x}_j)\mathbf{c}_j + \sum_{j=1}^{N} \Psi_{\text{curl}}(\mathbf{x}, \mathbf{x}_j)\mathbf{c}_j$, where $\mathbf{c}_j = a_j \boldsymbol{\mu}_j + b_j \boldsymbol{\zeta}_j$

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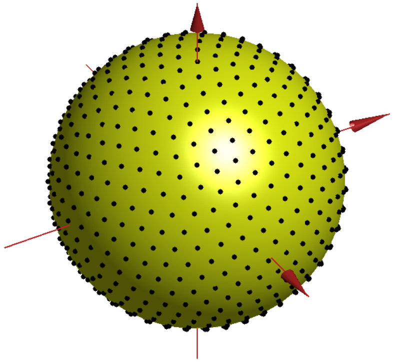

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If $\mathbf{u} \in H^{\beta}(\mathbb{S}^2)$, with $1 \leq \beta \leq \tau$ then we have

$$\|\mathbf{u} - \mathbf{s}\|_{L^2(\mathbb{S}^2)} \leq C \rho_X^{\tau - \beta} h_X^{\beta} \|\mathbf{u}\|_{H^{\beta}(\mathbb{S}^2)},$$

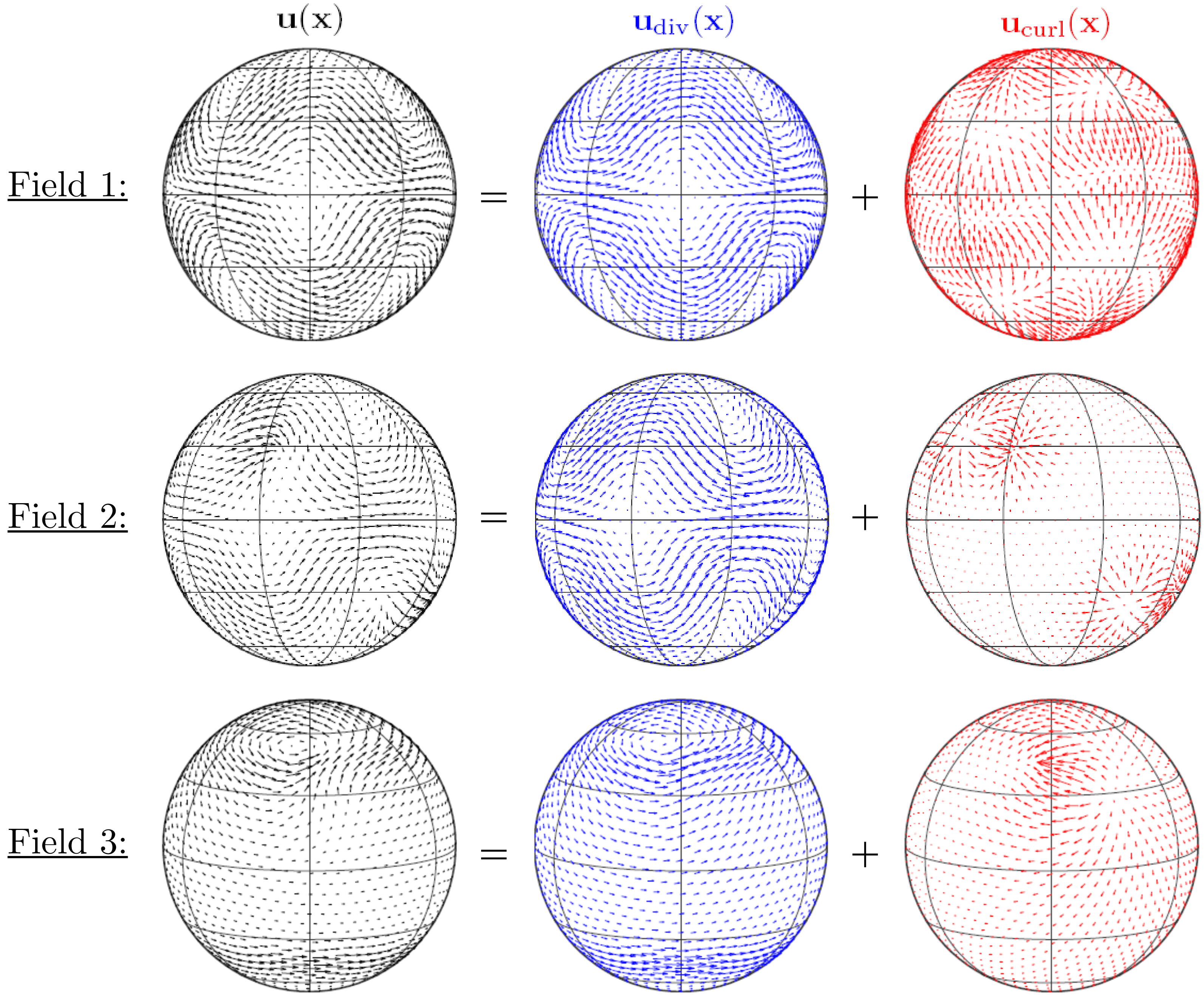
$$\|\mathbf{u}_{\text{div}} - \mathbf{s}_{\text{div}}\|_{L^2(\mathbb{S}^2)} \leq C \rho_X^{\tau - \beta} h_X^{\beta} \|\mathbf{u}\|_{H^{\beta}(\mathbb{S}^2)},$$

$$\|\mathbf{u}_{\text{curl}} - \mathbf{s}_{\text{curl}}\|_{L^2(\mathbb{S}^2)} \leq C \rho_X^{\tau - \beta} h_X^{\beta} \|\mathbf{u}\|_{H^{\beta}(\mathbb{S}^2)}.$$

Ex: Accuracy of reconstruction and decomposition

- Three fields of varying smoothness sampled at "scattered" nodes on the sphere:

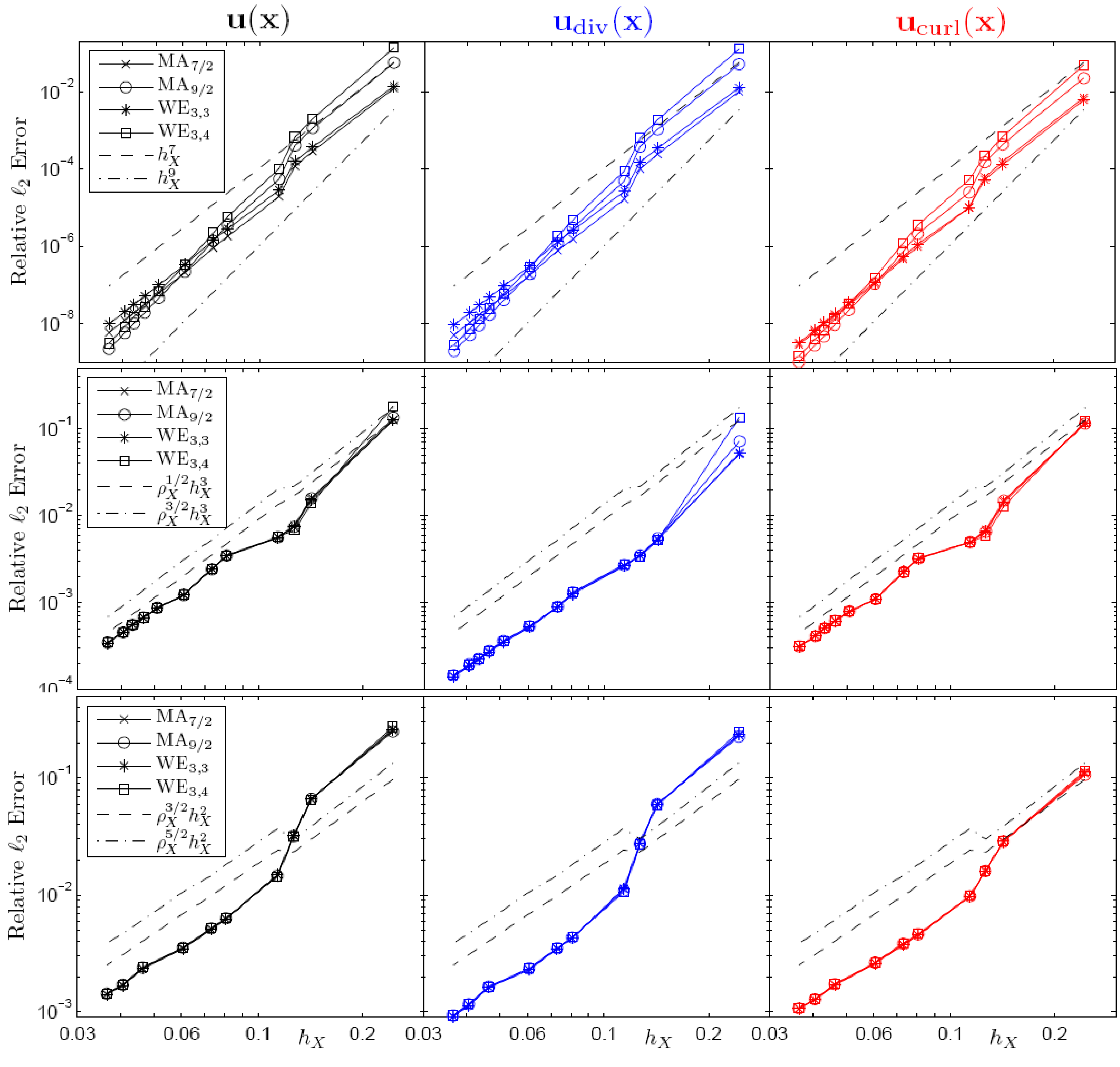
- Only the $\mathbf{u}(\mathbf{x})$ fields (first column) are sampled.



Ex: Accuracy of reconstruction and decomposition

- Error in the RBF reconstructed and decomposed field vs. node spacing (log-log scale):

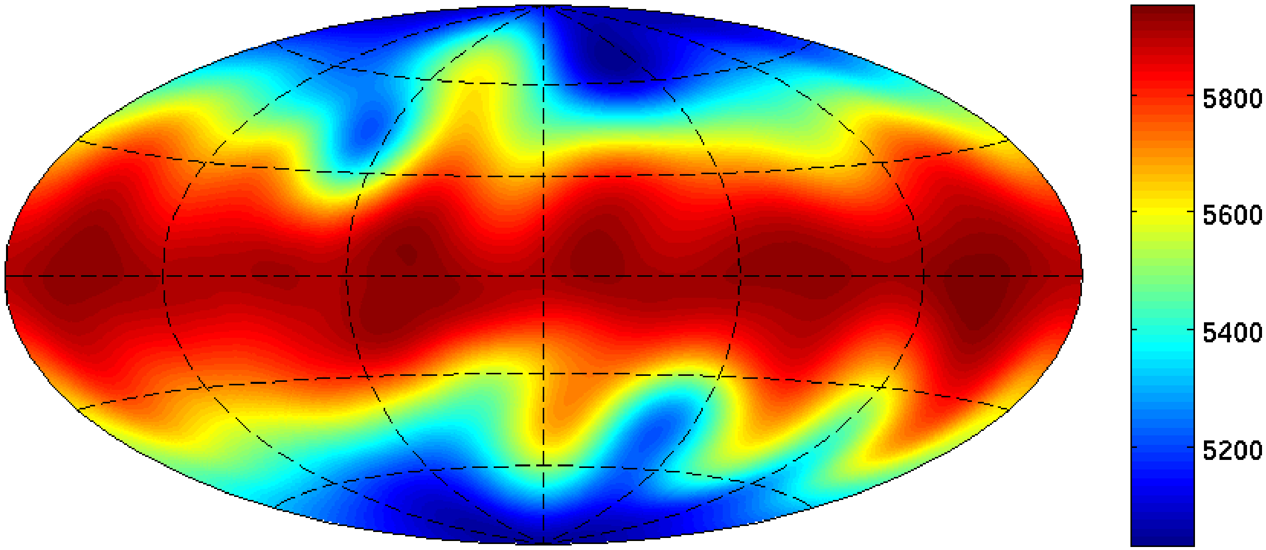
- Dashed lines correspond to predicted error rates from (Fuselier and Wright 2009)



Ex: decomposition of geophysical velocity field

- Numerical simulation of flow over an isolated mountain.

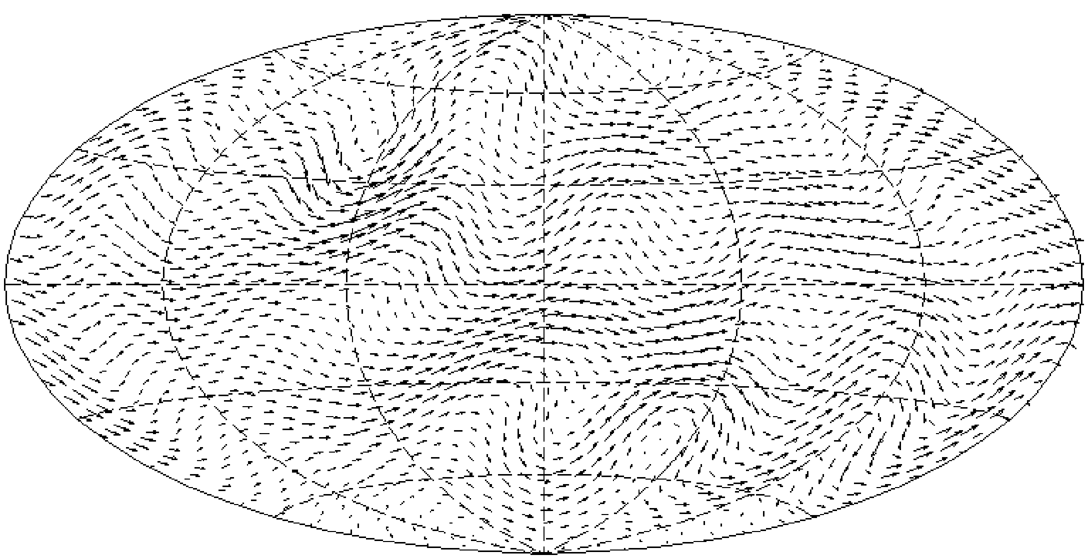
geopotential field t=15 days



Solution details

- GME SWM (Majewski *et. al.* MWR 2002)
- Icosahedral grid point model (92162 grid points).

Velocity field $\mathbf{u}$ t=15 days



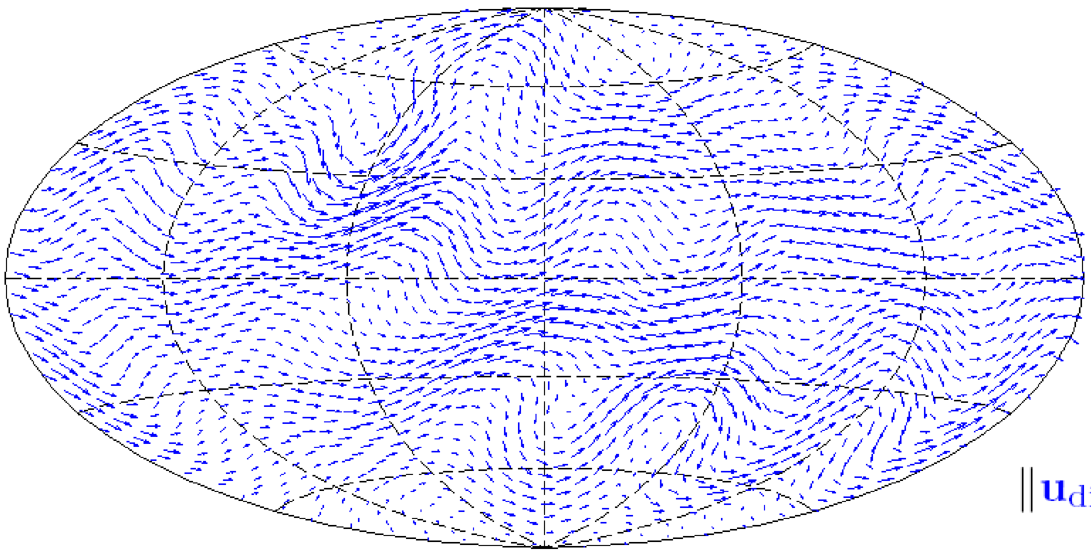
Sample details

- Sample velocity field of GME solution at N=1849 "scattered" nodes.
- Construct Helmholtz-Hodge RBF interpolant using Matérn $MA_{9/2}$ RBF

Ex: decomposition of geophysical velocity field

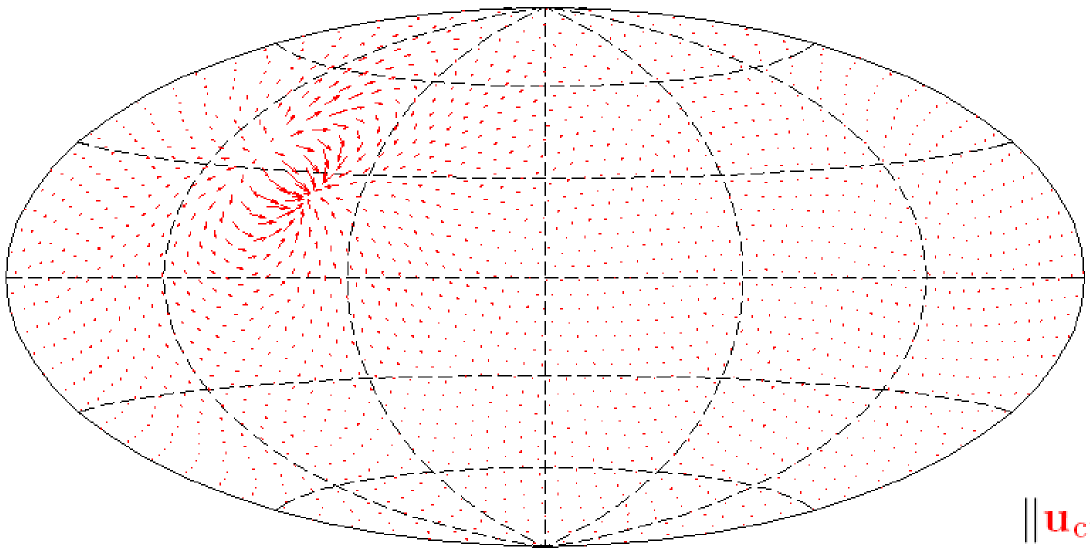
- Numerical simulation of flow over an isolated mountain.

RBF reconstructed div-free velocity field t=15 days



$\|\mathbf{u}_{\text{div}}\|_2 = 40.3$ m/s

RBF reconstructed curl-free velocity field t=15 days

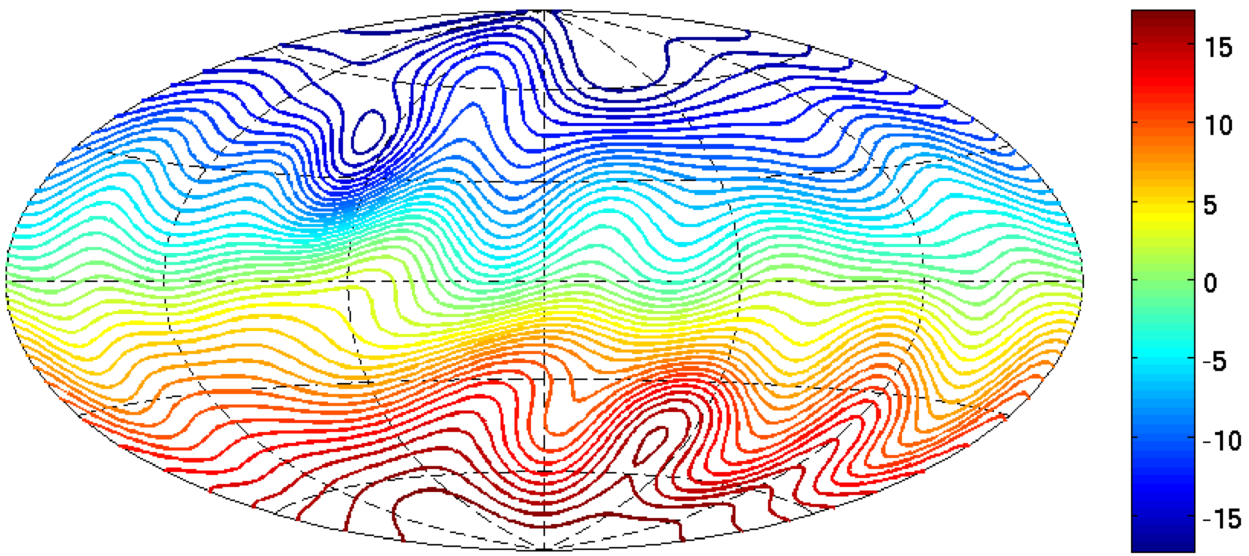


$\|\mathbf{u}_{\text{curl}}\|_2 = 4.3$ m/s

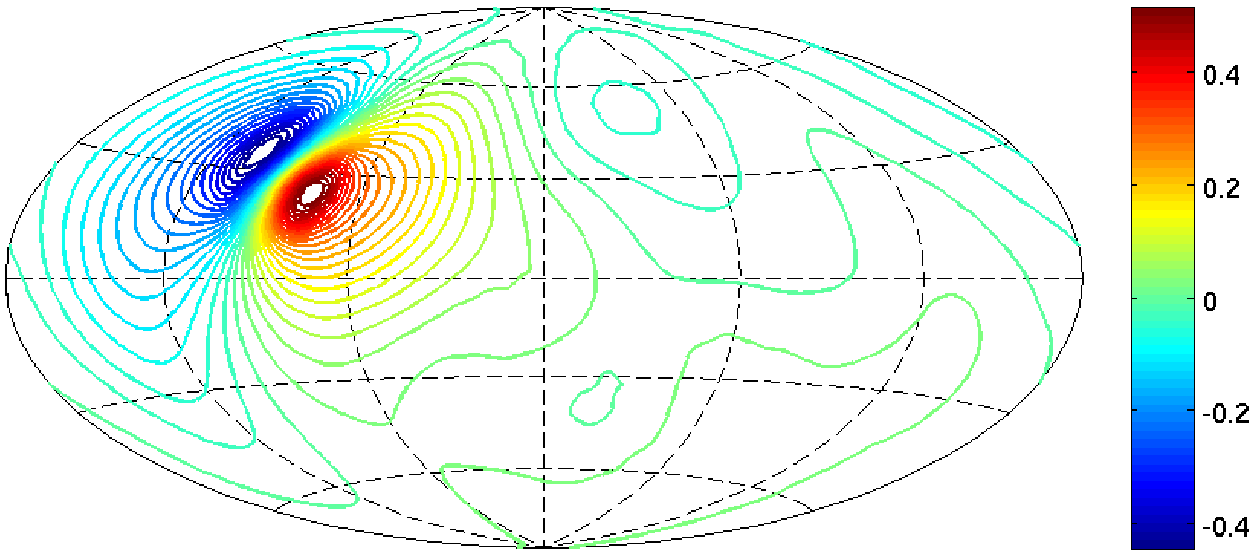
Ex: decomposition of geophysical velocity field

- Test case 5 (flow over an isolated mountain) from Williamson *et. al.* JCP (1992).

RBF reconstructed stream function t=15 days

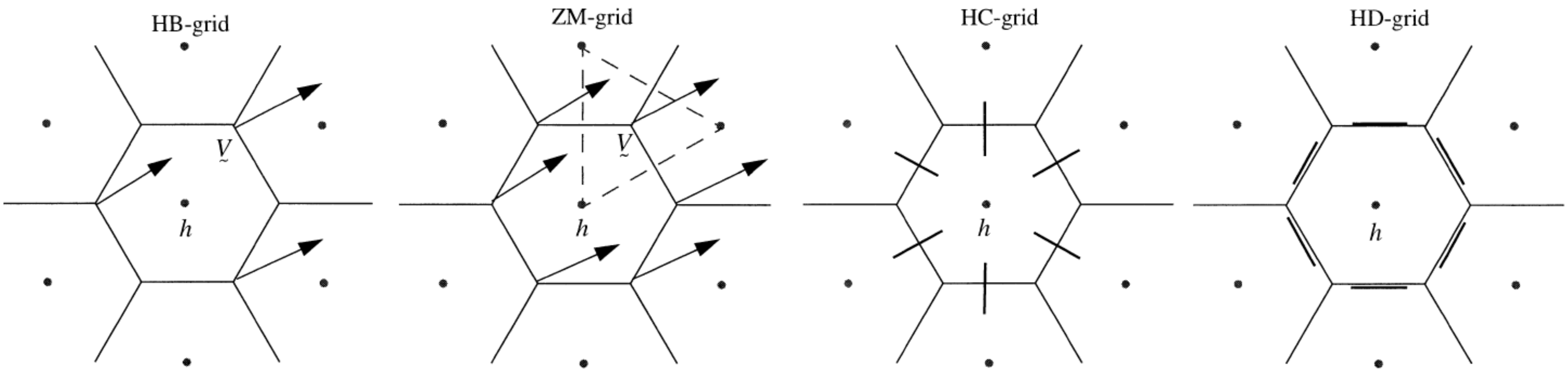


RBF reconstructed velocity potential t=15 days



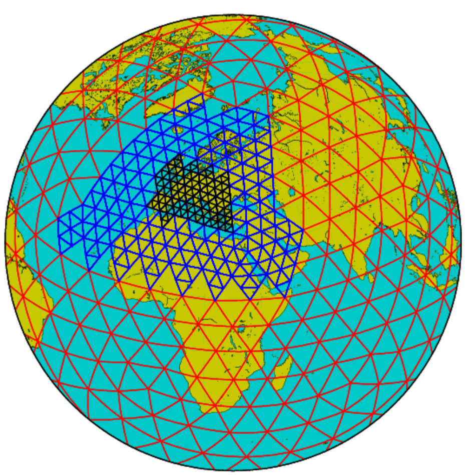
Local reconstruction of tangent vector fields

- Local reconstruction of velocity fields on staggered grids:

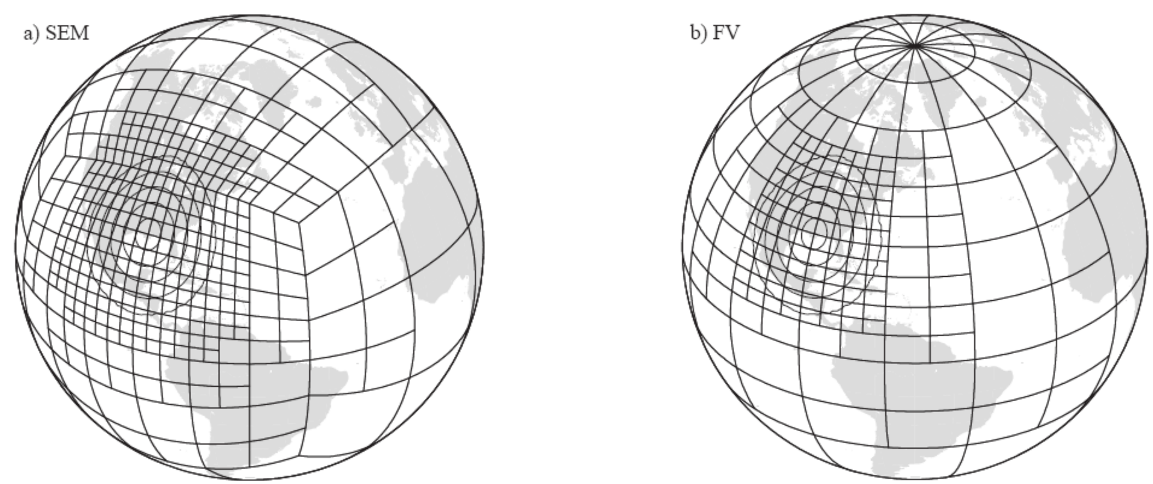


Ringler, T.D., and D.A. Randall, 2002: The ZM-grid: An alternative to the Z-grid, Monthly Weather Review,130, 1411-1422.

- Local reconstruction of velocity fields for adaptive mesh refinement (AMR)



http://icon.enes.org/



St-Cyr, A., C. Jablonowski, J. M. Dennis, H. M. Tufo and S. J. Thomas, A Comparison of Two Shallow Water Models with Non-Conforming Adaptive Grids, Mon. Wea. Rev., 136, 1898-1922, 2008.

Future directions

- Show connection between surface divergence and curl free RBF interpolants and divergence and curl free spherical harmonics.

- Develop numerical method for Navier Stokes equations on a rotating sphere:

Spherical coords:
$$\frac{\partial}{\partial t}\mathbf{u}_s + \nabla_s^* p = -\nabla_{\mathbf{u}_s}^* \mathbf{u}_s + \nu \Delta_s^* \mathbf{u}_s - f\hat{\mathbf{k}} \times \mathbf{u}_s + \mathbf{g}_s,$$
$$\nabla_s^* \cdot \mathbf{u}_s = 0,$$

Cartesian coords:
$$\frac{\partial}{\partial t}\mathbf{u} + P_{\mathbf{x}} \nabla p = -P_{\mathbf{x}}(((P_{\mathbf{x}}\nabla) \otimes \mathbf{u})^T \mathbf{u}) + \nu Q_{\mathbf{x}} \nabla(\mathbf{x} \cdot (\nabla \times \mathbf{u})) - f Q_{\mathbf{x}} \mathbf{u} + \mathbf{g},$$
$$\nabla \cdot \mathbf{u} = 0,$$

- Develop other decompositions (e.g. add radial component, toroidal/poloidal).

- Develop fast algorithms for computing and evaluating the interpolants.

- Extend to more general orientable manifolds:
  - Oblate/prolate spheroids and ellipsoids being the most applicable.

- Extend to bounded domains (work already under way).