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New family of Bernoulli–type polynomials and some application

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Abstract

In this paper, we present a new family of generalized Bernoulli–type polynomials, as well as its numbers. In addition, we obtain some results such as algebraic and differential properties for this new family of Bernoulli–type polynomials. Likewise, the generalized Bernoulli–type polynomials matrix $\mathscr{R}^{(\alpha)}(x)$ is introduced. We deduce some product formulae for $\mathscr{R}^{(\alpha)}(x)$ and also, the inverse of the Bernoulli–type matrix $\mathscr{R}$ is determined. Furthermore, we establish some explicit expressions for the Bernoulli–type polynomial matrix $\mathscr{R}(x)$, which involve the generalized Pascal matrix and finally we study the summation formula of Euler–Maclaurin type and the Riemann zeta function applied to these Bernoulli–type polynomials.

Keywords Bernoulli polynomials, generalized Bernoulli polynomials, Bernoulli polynomials matrix, Pascal matrix.
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1 Introduction

The Bernoulli polynomials, as well as the Bernoulli numbers have an important role in number theory and classical analysis. In particular, the Bernoulli polynomials appear in the integral representation of differentiable periodic functions since they are employed for approximating such functions in terms of polynomials, (see, [1, 2, 4, 5, 6, 9, 10, 13]). The classical Bernoulli polynomials $B_n(x)$, are defined by means of the following generating function (see, [9, p. 61])

$$\left(\frac{z}{e^z-1}\right)e^{zx}=\sum_{n=0}^{\infty}B_n(x)\frac{z^n}{n!},\quad |z|<2\pi. \tag{1}$$

For the classical Bernoulli numbers B_n, we readily find from (1) that

$$B_n:=B_n(0)=B_n^{(0)},\quad (n\in\mathbb{N}_0).$$

Numerous interesting properties involving these polynomials can be found in (for example, [8, 9, 12]).

Let $B_n(x)$ and B_n be the polynomials and numbers of Bernoulli, respectively. Then the following statements hold.

(a) Difference equation. [10, Equation (4)] For every $n\geq 0$

$$B_n(x+1)-B_n(x)=nx^{n-1}.$$

(b) Multiplication formula. [10, Equation (12)] For every $n\geq 0$ and $m\in\mathbb{N}$

$$B_n(mx)=m^{n-1}\sum_{k=0}^{m-1}B_n\left(x+\frac{k}{m}\right).$$

(c) Addition theorem of the argument. [10, Equation (13)] For $x,y\in\mathbb{R}$ and $n\in\mathbb{N}_0$, we have

$$B_n(x+y)=\sum_{k=0}^{n}B_k(x)y^{n-k}.$$

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(d) Integral formulas. [10, Equation (15)] For every $n \in \mathbb{N}$

$$\int_x^{x+1} B_n(t)dt = x^n.$$

In the present work we define a new family of generalized Bernoulli–type polynomials, and we study algebraic and differential properties. We also show some applications of this new family of Bernoulli–type polynomials. The paper is organized as follows. In Section 3, we define the new family of Bernoulli–type polynomials and Bernoulli–type numbers respectively, as well as the generalized Bernoulli–type polynomials; we prove some relevant algebraic and differential properties of them. In Section 4, we introduce some applications such as generalized Bernoulli–type polynomial matrix, we derive a product formula for it and give some factorizations for such a matrix, which involve summation matrices and the generalized Pascal matrix of the first kind. In addition, we will study the summation formula of Euler–Maclaurin type and the Riemann zeta function applied to Bernoulli–type polynomials.

2 Background and previous results

Throughout this paper, all matrices are in $M_{n+1}(\mathbb{R})$, the set of all $(n+1)$-square matrices over the real field. Also, for i, j any non negative integers we adopt the following convention

$$\binom{i}{j} = 0, \text{ whenever } j > i.$$

Let x be any nonzero real number. The generalized Pascal matrix of first kind $P[x]$ is an $(n+1) \times (n+1)$ matrix whose entries are given by (see [3, 15]):

$$p_{i,j}(x) = \begin{cases} \binom{i}{j} x^{i-j}, & i \geq j, \\ 0, & \text{otherwise}. \end{cases}$$

In [3, 15, 16] some properties of the generalized Pascal matrix of first kind are showed, for example, its matrix factorization by special summation matrices, its associated differential equation and its bivariate extensions. Let $P[x]$ be the generalized Pascal matrix of first kind and order $n+1$. Then the following statements hold.

(a) Special value. If the convention $0^0 = 1$ is adopted, then it is possible to define

$$P[0] := I_{n+1} = \operatorname{diag}(1, 1, \ldots, 1), \tag{2}$$

where I_{n+1} denotes the identity matrix of order $n+1$.

(b) $P[x]$ is an invertible matrix and its inverse is given by

$$P^{-1}[x] := (P[x])^{-1} = P[-x]. \tag{3}$$

(c) [3, Theorem 2] Addition theorem of the argument. For $x, y \in \mathbb{R}$ we have

$$P[x+y] = P[x]P[y].$$

(d) [3, Theorem 5] Differential relation (Appell type polynomial entries). $P[x]$ satisfies the following differential equation

$$D_x P[x] = \mathfrak{L} P[x] = P[x]\mathfrak{L},$$

where $D_x P[x]$ is the matrix resulting from taking the derivative with respect to x of each entry of $P[x]$ and the entries of the $(n+1) \times (n+1)$ matrix $\mathfrak{L}$ are given by

$$\mathrm{l}_{i,j} = \begin{cases} p'_{i,j}(0), & i \geq j, \\ 0, & \text{otherwise}, \end{cases}$$
$$= \begin{cases} j+1, & i = j+1, \\ 0, & \text{otherwise}. \end{cases}$$

(e) ([15, Theorem 1]). The matrix $P[x]$ can be factorized as follows.

$$P[x] = G_n[x]G_{n-1}[x] \cdots G_1[x],$$

where $G_k[x]$ is the $(n+1)\times(n+1)$ summation matrix given by

$$G_k[x]=\begin{cases}\begin{bmatrix} I_{n-k} & 0\\ 0 & S_k[x]\end{bmatrix}, & k=1,\dots,n-1,\\ \\ S_n[x], & k=n,\end{cases}$$

being $S_k[x]$ the $(k+1)\times(k+1)$ matrix with $(0\le i,j\le k)$ whose entries $S_k(x;i,j)$ are given by

$$S_k(x;i,j)=\begin{cases} x^{i-j}, & j\le i,\\ \\ 0, & j>i.\end{cases}$$

3 New family Bernoulli–type polynomials $R_n(x)$

In this section, we present some novel properties for a new family of Bernoulli–type polynomials.

Definition 3.1. The new family of Bernoulli–type polynomials $R_n(x)$ of degree n in x are defined by the generating function

$$\left(\frac{z^2}{2e^z-2}\right)e^{zx}=\sum_{n=0}^{\infty}R_n(x)\frac{z^n}{n!},\quad |z|<2\pi. \tag{4}$$

The first six Bernoulli–type polynomials, $R_n(x)$, are

$R_0(x)=0,$

$R_1(x)=\frac{1}{2},$

$R_2(x)=x-\frac{1}{2},$

$R_3(x)=\frac{3}{2}x^2-\frac{3}{2}x+\frac{1}{4},$

$R_4(x)=2x^3-3x^2+x,$

$R_5(x)=\frac{5}{2}x^4-5x^3+\frac{5}{2}x^2-\frac{1}{12}.$

For $x=0$ in (4) the Bernoulli–type numbers are defined by the generating function

$$\frac{z^2}{2e^z-2}=\sum_{n=0}^{\infty}\frac{R_n z^n}{n!},\quad |z|<2\pi. \tag{5}$$

Some of these numbers are

$$R_0=0;\quad R_1=\frac{1}{2};\quad R_2=-\frac{1}{2};\quad R_3=\frac{1}{4};\quad R_4=0;\quad R_5=-\frac{1}{12};\quad R_6=0;\quad R_7=-\frac{1}{12};\quad R_8=0.$$

A consequence of (4) and (5) is the following proposition.

Proposition 3.1. *For $n\in\mathbb{N}$, let $\{R_n(x)\}_{n\ge0}$ be the sequences of Bernoulli–type polynomials in the variable x. Then the following statements hold.*

(i) $R_n(x)=\sum_{k=0}^{n}\binom{n}{k}R_k x^{n-k}.$

(ii) $R_n(x+1)-R_n(x)=\frac{n(n-1)}{2}x^{n-2}.$

(iii) $R'_{n+1}(x)=(n+1)R_n(x).\quad n\in\mathbb{N},$

(iv) $\displaystyle\frac{2}{n+1}\int_1^{m+1}R_{n+1}(t)dt=\sum_{k=1}^{m}k^n,\quad n\in\mathbb{N}.$

(v) $\displaystyle\int_x^{x+1}R_n(t)dt=\frac{n}{2}x^{n-1},\quad n\in\mathbb{N}.$

(vi) $\displaystyle\int_x^{y}R_n(t)dt=\frac{R_{n+1}(y)-R_{n+1}(x)}{(n+1)},\quad n\in\mathbb{N}_0-\{1\}.$

Proof. For the proof of (i), by multiplying both sides of (5) by e^{xz}, from (4)

$$\begin{aligned}\frac{z^2}{2e^z-2}e^{xz} &= \sum_{n=0}^{\infty}R_n\frac{z^n}{n!}e^{xz}\\ \sum_{n=0}^{\infty}R_n(x)\frac{z^n}{n!} &= \sum_{n=0}^{\infty}R_n\frac{z^n}{n!}\sum_{n=0}^{\infty}x^n\frac{z^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}R_k x^{n-k}\frac{z^n}{n!}.\end{aligned}$$

Equating the coefficients of $\frac{z^n}{n!}$, we obtain the desired result. □

Proof. For the proof of (ii), consider the following generating functions

$$\frac{z^2}{2e^z-2}e^{(x+1)z}=\sum_{n=0}^{\infty}R_n(x+1)\frac{z^n}{n!}, \tag{6}$$

and

$$\frac{z^2}{2e^z-2}e^{xz}=\sum_{n=0}^{\infty}R_n(x)\frac{z^n}{n!}. \tag{7}$$

Subtracting the left-hand sides of (6) and (7), we have

$$\begin{aligned}\frac{z^2}{2e^z-2}e^{(x+1)z}-\frac{z^2}{2e^z-2}e^{xz} &= \frac{z^2}{2e^z-2}e^{xz}(e^z-1)\\ &= \frac{z^2}{2}e^{xz}\\ &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^{n+2}x^n}{n!}\\ &= \frac{1}{2}\sum_{n=2}^{\infty}n(n-1)x^{n-2}\frac{z^n}{n!}.\end{aligned}$$

Then,

$$\sum_{n=0}^{\infty}[R_n(x+1)-R_n(x)]\frac{z^n}{n!}=\sum_{n=2}^{\infty}\frac{n(n-1)}{2}x^{n-2}\frac{z^n}{n!}.$$

Equating the coefficients of $\frac{z^n}{n!}$, we obtain the result. □

The proof of (iii) is omitted as it is well-known that any family of polynomials of Appell type satisfies this property. The proof of (iv), (v) and (vi) follow a similar approach to those employed in Theorem 4 in [11, p. 293].

Theorem 3.2. *The Bernoulli–type numbers are equal to zero for all integer greater than two, that is*

$$R_n=0 \qquad \textit{for all} \quad n\geq 4.$$

Proof. From the generating function (5), we have

$$\frac{z^2}{2e^z-2}=\sum_{n=0}^{\infty}R_n\frac{z^n}{n!}=R_0+R_1z+\frac{R_2}{2!}z^2+\frac{R_3}{3!}z^3+\sum_{n=4}^{\infty}R_n\frac{z^n}{n!}. \tag{8}$$

Solving for (8) and replacing them with the numbers R_0, R_1, R_2 and R_3, we have

$$\sum_{n=0}^{\infty}R_n\frac{z^n}{n!}=\frac{z^2}{2e^z-2}-\frac{1}{2}z+\frac{1}{4}z^2-\frac{1}{24}z^3.$$

Consider the following function:

$$f(z)=\frac{z^2}{2e^z-2}-\frac{1}{2}z+\frac{1}{4}z^2-\frac{1}{24}z^3. \tag{9}$$

By performing some calculations, it can be seen that (9) is an odd function. So that

$$f(-z)=\sum_{n=4}^{\infty}R_n\frac{(-z)^n}{n!}=-\sum_{n=4}^{\infty}R_n\frac{z^n}{n!}, \tag{10}$$

and

$$f(-z)=\sum_{n=4}^{\infty}R_n\frac{(-z)^n}{n!}=\sum_{n=4}^{\infty}(-1)^nR_n\frac{z^n}{n!}. \tag{11}$$

Therefore, from (10) and (11), we get

$$-R_n=(-1)^nR_n, \quad \forall n\geq 4.$$

Thus if n is even, then $-R_n=R_n$. This proves the Theorem. □

Lemma 3.3. *For $N, M\in\mathbb{N}$ with $N>M$ and $0\leq j\leq k-1$, we have*

$$\sum_{q=M}^{N-1}\sum_{j=0}^{k-1}(kq+j)^n=\sum_{m=Mk}^{Nk-1}m^n=\frac{2}{(n+2)(n+1)}\{\underbrace{R_{n+2}[(Nk-1)+1]}_{R_{n+2}(Nk)}-R_{n+2}(Mk).\}$$

Proof.

$$\begin{aligned}
\sum_{q=M}^{N-1}\sum_{j=0}^{k-1}(kq+j)^n &= \sum_{q=M}^{N-1}[(kq)^n+(kq+1)^n+(kq+2)^n+\cdots+(kq+k-1)^n]\\
&= [(kM)^n+(kM+1)^n+(kM+2)^n+\cdots+(kM+k-1)^n]\\
&\quad +[(kM+k)^n+(kM+k+1)^n+(kM+k+2)^n+\cdots+(kM+2k-1)^n]\\
&\quad +[(kM+2k)^n+(kM+2k+1)^n+(kM+2k+2)^n+\cdots+(kM+3k-1)^n]\\
&\quad \vdots\\
&\quad +[(kN-k)^n+(kN-k+1)^n+(kN-k+2)^n+\cdots+(kN-1)^n]\\
&= \sum_{m=Mk}^{Nk-1} m^n = \frac{2}{(n+2)(n+1)}\{\underbrace{R_{n+2}[(Nk-1)+1]}_{R_{n+2}(Nk)}-R_{n+2}(Mk).\}
\end{aligned}$$

So we have reached the testing phase. □

Theorem 3.4. *(Multiplication formula). The Bernoulli–type polynomials $R_n(x)$, of variable x, satisfies the following relation*

$$R_{n+1}(mx)=m^{n-1}\sum_{k=0}^{m-1}R_{n+1}(x+\frac{k}{m}),\quad n\in\mathbb{N}_0,\quad m\in\mathbb{N}.$$

Proof. From (ii) of Proposition 3.1, we get,

$$(kq+j)^n=\frac{2}{(n+2)(n+1)}k^n\left[R_{n+2}(q+\frac{j}{k}+1)-R_{n+2}(q+\frac{j}{k})\right],\quad k\neq 0,$$

If $N,M\in\mathbb{N}$ with $N>M$ and $0\leq j\leq k-1$, we have

$$\begin{aligned}
\sum_{q=M}^{N-1}\sum_{j=0}^{k-1}(kq+j)^n &= \frac{2}{(n+2)(n+1)}k^n\sum_{q=M}^{N-1}\sum_{j=0}^{k-1}\left[R_{n+2}(q+\frac{j}{k}+1)-R_{n+2}(q+\frac{j}{k})\right]\\
&= \frac{2k^n}{(n+2)(n+1)}\sum_{j=0}^{k-1}\left[R_{n+2}(N+\frac{j}{k})-R_{n+2}(M+\frac{j}{k})\right].
\end{aligned}$$

From the Lemma 3.3, we have

$$\sum_{q=M}^{N-1}\sum_{j=0}^{k-1}(kq+j)^n=\sum_{m=Mk}^{Nk-1}m^n=\frac{2}{(n+2)(n+1)}\{\underbrace{R_{n+2}[(Nk-1)+1]}_{R_{n+2}(Nk)}-R_{n+2}(Mk)\},$$

then

$$[R_{n+2}(Nk)-R_{n+2}(Mk)]=k^n\sum_{j=0}^{k-1}\left[R_{n+2}(N+\frac{j}{k})-R_{n+2}(M+\frac{j}{k})\right].$$

So,

$$R_{n+2}(Nk)-k^n\sum_{j=0}^{k-1}R_{n+2}(N+\frac{j}{k})=R_{n+2}(Mk)-k^n\sum_{j=0}^{k-1}R_{n+2}(M+\frac{j}{k}),\quad \forall N>M.$$

On the other hand, consider the following function

$$f(x)=R_{n+2}(xk)-k^n\sum_{j=0}^{k-1}R_{n+2}(x+\frac{j}{k}).$$

For a fixed $M\in\mathbb{N}$, we have

$$f(M+1)=f(M+2)=f(M+3)=\cdots.$$

Therefore $f(x)$ must be a constant polynomial and when deriving it, we will have

$$f'(x)=kR'_{n+2}(xk)-k^n\sum_{j=0}^{k-1}R'_{n+2}(x+\frac{j}{k})=0,$$

then

$$kR_{n+1}(xk)-k^n\sum_{j=0}^{k-1}R_{n+1}(x+\frac{j}{k})=0,$$

therefore

$$R_{n+1}(xk) = k^{n-1}\sum_{j=0}^{k-1} R_{n+1}(x+\frac{j}{k}).$$

Thus the thesis follows. □

Definition 3.2. For a real or complex parameter α, the generalized Bernoulli–type polynomials $R_n^{(\alpha)}(x)$, of degree n in x, are defined by means of the following generating functions

$$\left(\frac{z^2}{2e^z-2}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} R_n^{(\alpha)}(x)\frac{z^n}{n!}, \quad |z|<2\pi, \quad 1^{\alpha} := 1, \tag{12}$$

$$R_n(x) := R_n^{(1)}(x), \quad n \in \mathbb{N}_0,$$

where $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$ and $\mathbb{N} = \{1,2,3,\ldots\}$.

From the generating relation (12), it is fairly straightforward to deduce the addition formula:

$$R_n^{(\alpha+\beta)}(x+y) = \sum_{k=0}^{n}\binom{n}{k} R_k^{(\alpha)}(x) R_{n-k}^{(\beta)}(y). \tag{13}$$

Making an adequate substitution (13), we get

$$R_n^{(\alpha)}(x+y) = \sum_{k=0}^{n}\binom{n}{k} R_k^{(\alpha)}(y) x^{n-k}.$$

As an immediate consequence (13), we have

$$R_n(x+y) = \sum_{k=0}^{n}\binom{n}{k} R_k(y) x^{n-k},$$

$$R_n(x) = \sum_{k=0}^{n}\binom{n}{k} R_k\, x^{n-k},$$

$$R_n^{(\alpha)}(x) = 0, \quad n<\alpha.$$

4 Some applications

Inspired by the article [14] in which the authors introduce the generalized Apostol-type polynomial matrix, in this section we focus our attention on the algebraic and differential properties of the Bernoulli–type polynomial matrix. We will also show summation formula of Euler-Maclaurin type based on the Bernoulli–type polynomials and the values of the Riemann zeta function for Bernoulli–type numbers.

Definition 4.1. The generalized $(n+1)\times(n+1)$ Bernoulli–type polynomial matrix $\mathscr{R}^{(\alpha)}(x)$ is defined by

$$R_{i,j}^{(\alpha)}(x) = \begin{cases} \dfrac{\binom{i+1}{j+1}}{\alpha!\binom{i-j+\alpha}{\alpha}} R_{i-j+\alpha}^{(\alpha)}(x), & i \geq j, \\ 0, \quad \text{otherwise}, \end{cases}$$

while, $\mathscr{R}(x) := \mathscr{R}^{(1)}(x)$ and $\mathscr{R} := \mathscr{R}(0)$ are called Bernoulli–type polynomial matrix and Bernoulli–type matrix, respectively.

Let us consider $n=3$. It follows from the Definition 4.1 that

$$\mathscr{R} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad \mathscr{R}(x) = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ x-\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{3}{2}x^2-3x+\frac{1}{4} & \frac{3}{2}x-\frac{3}{2} & \frac{1}{2} & 0 \\ 2x^3-3x^2+x & 3x^2-3x+\frac{1}{2} & 2x-1 & \frac{1}{2} \end{bmatrix}.$$

Theorem 4.1. *The generalized Bernoulli–type polynomial matrix $\mathscr{R}^{(\alpha)}(x)$ satisfies the following product formula.*

$$\mathscr{R}^{(\alpha+\beta)}(x+y) = \mathscr{R}^{(\alpha)}(x)\mathscr{R}^{(\beta)}(y) = \mathscr{R}^{(\beta)}(x)\mathscr{R}^{(\alpha)}(y) = \mathscr{R}^{(\alpha)}(y)\mathscr{R}^{(\beta)}(x). \tag{14}$$

Proof. Let $W_{i,j}^{(\alpha,\beta)}(x,y)$. Be the (i,j)-th entry of the matrix product $\mathscr{R}^{(\alpha)}(x)\mathscr{R}^{(\beta)}(y)$. By the addition formula (13), we have.

$$W_{i,j}^{(\alpha,\beta)}(x,y) = \sum_{k=j}^{i} \frac{\binom{i+1}{k+1}}{\alpha!\binom{i-k+\alpha}{\alpha}} R_{i-k+\alpha}^{(\alpha)}(x) \frac{\binom{k+1}{j+1}}{\beta!\binom{k-j+\beta}{\beta}} R_{k-j+\beta}^{(\beta)}(y)$$

$$= \frac{\binom{i+1}{j+1}}{(\alpha+\beta)!\binom{i-j+\alpha+\beta}{\alpha+\beta}} \sum_{k=j}^{i} \frac{(i-j+\alpha+\beta)!}{(i-k+\alpha)!(k-j+\beta)!} R_{i-k+\alpha}^{(\alpha)}(x) R_{k-j+\beta}^{(\beta)}(y)$$

$$= \frac{\binom{i+1}{j+1}}{(\alpha+\beta)!\binom{i-j+\alpha+\beta}{\alpha+\beta}} \sum_{k=\alpha+\beta}^{i-j+\alpha+\beta} \binom{i-j+\alpha+\beta}{k-\alpha} R_{i-j+\alpha+\beta-(k-\alpha)}^{(\alpha)}(x) R_{k-\alpha}^{\beta}(y)$$

$$= \frac{\binom{i+1}{j+1}}{(\alpha+\beta)!\binom{i-j+\alpha+\beta}{\alpha+\beta}} R_{i-j+\alpha+\beta}^{(\alpha+\beta)}(x+y),$$

which implies the first equality of (14). The second and third equalities of (14) can be derived in a similar way. □

Corollary 4.2. *The generalized Bernoulli–type polynomial matrix $\mathscr{R}^{(\alpha)}(x)$ satisfies the following identity.*

$$\left(\mathscr{R}^{(\alpha)}(x)\right)^k = \mathscr{R}^{(k\alpha)}(kx).$$

In particular,

$$\begin{aligned} (\mathscr{R}(x))^k &= \mathscr{R}^{(k)}(kx), \\ \mathscr{R}^k &= \mathscr{R}^{(k)}. \end{aligned}$$

Let be the $(n+1)\times(n+1)$ matrix whose entries are defined by

$$f_{i,j} = \begin{cases} \frac{2}{i-j+1}\binom{i+1}{j+1}, & i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4.3. *The inverse Bernoulli–type matrix $\mathscr{R}$ is given by*

$$\mathscr{R}^{-1} = .$$

Proof. Given

$$\sum_{k=0}^{n} \frac{2}{(k+1)(n-k+1)} \binom{n}{k} R_{n-k+1} = \delta_{n,0},$$

where $\delta_{n,0}$ is the Kronecker delta (cf, [17]), we have

$$\sum_{k=j}^{i} \frac{\binom{i+1}{k+1}}{i-k+1} R_{i-k+1} \frac{2}{k-j+1}\binom{k+1}{j+1} = \binom{i+1}{j+1} \sum_{k=j}^{i} \binom{i-j}{k-j} R_{i-k+1} \frac{2}{k-j+1} \frac{1}{i-k+j}$$

$$= \binom{i+1}{j+1} \sum_{k=0}^{i-j} \binom{i-j}{k} R_{i-k-j+1} \frac{2}{k+1} \frac{1}{i-k-j+1}$$

$$= \binom{i+1}{j+1} \delta_{i-j}, 0.$$

The proof is finished. □

The next result establishes the relationship between the Bernoulli–type polynomial matrix and the generalized Pascal matrix of first kind.

Theorem 4.4. *The Bernoulli–type polynomial matrix $\mathscr{R}(x)$ satisfies the following relation.*

$$\mathscr{R}(x+y) = P[x]\mathscr{R}(y) = P[y]\mathscr{R}(x), \tag{15}$$
$$\mathscr{R}'(x) = P[x]\mathscr{R}.$$

Proof. The substitution $\beta = 0$ into (14) yields

$$\mathscr{R}^{(\alpha)}(x+y) = \mathscr{R}^{(\alpha)}(x)\mathscr{R}^{(0)}(y) = \mathscr{R}^{(0)}(x)\mathscr{R}^{(\alpha)}(y) = \mathscr{R}^{(\alpha)}(y)\mathscr{R}^{(0)}(x).$$

Since $\mathscr{R}^{(0)}(x) = P[x]$, we obtain

$$\mathscr{R}^{(\alpha)}(x+y) = P[x]\mathscr{R}^{(\alpha)}(y). \tag{16}$$

Next, the substitution $\alpha = 1$ into (16) yields (15). □

Theorem 4.5. *(Summation formula of Euler-Maclaurin type based on the Bernoulli–type polynomials $R_n(x)$). Let $\overline{R_n}(x) = R_n(x - \lfloor x \rfloor)$ and suppose $f \in C^{k+1}[a,b]$ with $a, b \in \mathbb{Z}$. Then*

$$\sum_{a<n\leq b} f(n) = \int_a^b f(t)dt + 2\sum_{r=0}^{k} \frac{(-1)^{r+1}}{(r+2)!}\left[f^{(r)}(b) - f^{(r)}(a)\right]R_{r+2} + P_k, \tag{17}$$

where

$$P_k = \frac{2(-1)^k}{(k+2)!}\int_a^b \overline{R_{k+2}}(t) f^{(k+1)}(t)dt.$$

Proof. Consider $\int_n^{n+1} f(t)dt = \int_n^{n+1} f(t)\overline{R}_2'(t)dt$. Solving the integral in the right side by parts, we have $u = f(t) \Rightarrow du = f'(t)dt$, $dv = \overline{R}_2'(t)dt \Rightarrow v = \overline{R}_2(t)$

$$\begin{aligned}
\int_n^{n+1} f(t)dt = \int_n^{n+1} f(t)R_2'(t)dt &= f(t)\overline{R}_2(t)|_n^{n+1} - \int_n^{n+1}\overline{R}_2(t)f'(t)dt \\
&= f(n+1)\overline{R}_2[(n+1)^-] - f(n)\overline{R}_2(n^+) - \int_n^{n+1}\overline{R}_2(t)f'(t)dt \\
&= \frac{1}{2}f(n+1) + \frac{1}{2}f(n) - \int_n^{n+1}\overline{R}_2(t)f'(t)dt \\
&= \frac{f(n)+f(n+1)}{2} - \int_n^{n+1}\overline{R}_2(t)f'(t)dt,
\end{aligned}$$

where $\overline{R}_2[(n+1)^-] = \lim_{t\to(n+1)^-}\overline{R}_2(t) = \frac{1}{2}$ and $\overline{R}_2[(n)^+] = \lim_{t\to(n)^+}\overline{R}_2(t) = -\frac{1}{2}$.
So,

$$\int_n^{n+1} f(t)dt = \frac{f(n)+f(n+1)}{2} - \int_n^{n+1}\overline{R}_2(t)f'(t)dt. \tag{18}$$

Adding in (18) for $n = a, a+1, a+2, ..., b-1$; we obtain:

$$\begin{aligned}
\int_a^b f(t)dt &= \sum_{n=a}^{b-1}\frac{f(n)+f(n+1)}{2} - \int_a^b \overline{R}_2(t)f'(t)dt \\
&= \frac{f(a)+f(b)}{2} + \sum_{n=a+1}^{b-1} f(n) - \int_a^b \overline{R}_2(t)f'(t)dt.
\end{aligned}$$

Then,

$$\sum_{a<n\leq b} f(n) = \int_a^b f(t)dt - [f(b)-f(a)](-\frac{1}{2}) + \int_a^b \overline{R}_2(t)f'(t)dt.$$

This is the case for $k = 0$ in (17).

Suppose this theorem is true for $k = q$, that is to say

$$\sum_{a<n\leq b} f(n) = \int_a^b f(x)dx + 2\sum_{r=0}^{q}\frac{(-1)^{r+1}}{(r+2)!}[f^{(r)}(b) - f^{(r)}(a)]R_{r+2} + \frac{(-1)^q}{(q+2)!}2\int_a^b \overline{R}_{q+2}(t)f^{(q+1)}(t)dt. \tag{19}$$

Let's prove that it's true for $k=q+1$.

Adding and subtracting $\frac{(-1)^{q+2}}{(q+3)!}2[f^{(q+1)}(b)-f^{(q+1)}(a)]R_{q+3}$ on the right side of (19), we get

$$\sum_{a<n\leq b} f(n)=\int_a^b f(t)dt+2\sum_{r=0}^{q+1}\frac{(-1)^{r+1}}{(r+2)!}[f^{(r)}(b)-f^{(r)}(a)]R_{r+2}+\frac{(-1)^{q+1}}{(q+3)!}2\overline{R}_{q+3}(t)f^{(q+1)}(t)|_a^b-\frac{(-1)^{q+1}}{(q+3)!}2(q+3)\int_a^b \overline{R}_{q+2}(t)f^{(q+1)}(t)dt$$

$$=\int_a^b f(t)dt+2\sum_{r=0}^{q+1}\frac{(-1)^{r+1}}{(r+2)!}[f^{(r)}(b)-f^{(r)}(a)]R_{r+2}+\frac{(-1)^{q+1}}{(q+3)!}2\int_a^b \overline{R}_{q+3}(t)f^{(q+2)}(t)dt.$$

Which amounts to the theorem for $k=q+1$.

□

Example 4.1. Let $f(t)=\frac{1}{t}$; *and* $a=1$; $b=x$; $k=2$. From (17),

$$\begin{aligned}\sum_{1<n\leq x}^{x}\frac{1}{n} &= \int_1^x \frac{1}{t}dt+2\sum_{r=0}^{2}\frac{(-1)^{r+1}}{(r+2)!}\left[f^{(r)}(x)-f^{(r)}(1)\right]R_{r+2}+\frac{2}{4!}\int_1^x \overline{R}_4(t)f^{(3)}(t)dt\\ &= \ln|x|+2\left[-\frac{1}{2}\left(\frac{1}{x}-1\right)R_2+\frac{1}{6}\left(-\frac{1}{x^2}+1\right)R_3-\frac{1}{24}\left(\frac{2}{x^3}-2\right)R_4\right]-\frac{1}{2}\int_1^2\frac{\overline{R}_4(t)}{t^4}dt,\end{aligned}$$

then

$$\sum_{n=1}^{x}\frac{1}{n}-\ln|x|=\frac{1}{2x}+\frac{1}{2}-\frac{1}{12x^2}+\frac{1}{12}-\frac{1}{2}\int_1^x\frac{\overline{R}_4(t)}{t^4}dt. \tag{20}$$

Taking limit $x\to\infty$, on both sides, we have

$$\gamma=\frac{1}{2}+\frac{1}{12}-\frac{1}{2}\int_1^\infty\frac{\overline{R}_4(t)}{t^4}dt, \tag{21}$$

where γ is the constant of Euler-Masheroni (see, [7, p. 119, Eq (27a)]). From (20) and (21), we obtain

$$\sum_{n=1}^{x}\frac{1}{n}=\ln|x|+\frac{1}{2x}-\frac{1}{12x^2}+\gamma+O\left(\frac{1}{x^3}\right).$$

Theorem 4.6. *The values of the Riemann zeta function are given by (see [9])*

$$\zeta(k)=\sum_{n=1}^{\infty}\frac{1}{n^k}=(-1)^{\frac{k}{2}+1}(2\pi)^k\frac{R_{k+1}}{(k+1)!}$$

for k even, where R_{k+1} are the Bernoulli–type numbers defined (5).

Proof. Let's consider the function

$$f(z)=\pi\cot(\pi z)=\pi\frac{\cos(\pi z)}{\sin(\pi z)}$$

and the contour C_N with $N\in\mathbb{N}$ in the complex plane that consists in the edge of the square with vertices $(N+\frac{1}{2})(\pm1\pm i)$. Let

$$g(z)=\frac{1}{z^k}f(z) \tag{22}$$

with k even. It has simple poles in the integers $z=n$ with $n\neq 0$, and reminder $\frac{1}{n^k}$, because given $n_0\leq N$

$$\begin{aligned}\lim_{z\to n_0}\frac{(z-n_0)}{z^k}\pi\frac{\cos(\pi z)}{\sin(\pi z)} &= \lim_{z\to n_0}\frac{(\pi z-n_0\pi)}{z^k}\frac{\cos(\pi z)}{(-1)^{n_0}\sin(\pi z-n_0\pi)}\\ &= \frac{1}{n_0^k}.\end{aligned}$$

Then by the Residue Theorem

$$\int_{C_N} g(z)dz=2\pi i\left[\sum_{n=-N,n\neq 0}^{N}\frac{1}{n^k}+Res_{z=0}g(z)\right].$$

As k is an even number

$$\sum_{n=-N}^{N}\frac{1}{n^k}=2\sum_{n=1}^{N}\frac{1}{n^k},$$

so

$$\int_{C_N} g(z)dz = 2\pi i\left[2\sum_{n=1}^{N}\frac{1}{n^k} + Res_{z=0}g(z)\right]. \tag{23}$$

On the other hand, as $f(z)$ is bounded on the contour C_N, this means

$$|f(z)| \leq A \quad \text{for every } z \in C_N \text{ (with } A \text{ independent of } N\text{)}.$$

We get

$$\left|\int_{C_N} g(z)dz\right| = \left|\int_{C_N}\frac{1}{z^k}f(z)dz\right| \leq \int_{C_N}\left|\frac{1}{z^k}f(z)dz\right| \leq \int_{C_N}\left|\frac{1}{z^k}\right||f(z)|\,|dz|$$
$$\leq \frac{1}{\left(N+\frac{1}{2}\right)^k}A\cdot 4(2N+1) \to 0 \text{ as } N\to\infty.$$

Taking the limit (23) when $N\to\infty$, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{n^k} = -\frac{1}{2}Res_{z=0}g(z). \tag{24}$$

Now,we look for $Res_{z=0}g(z)$.
We obtain that

$$\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)} = i\frac{e^{\pi iz}+e^{-\pi iz}}{e^{\pi iz}-e^{-\pi iz}} = i\coth(\pi iz). \tag{25}$$

On the other side,

$$\begin{aligned}\coth\left(\frac{z}{2}\right) &= \frac{e^{\frac{z}{2}}+e^{-\frac{z}{2}}}{e^{\frac{z}{2}}-e^{-\frac{z}{2}}} = \frac{2z^2}{2z^2}\left(\frac{e^z}{e^z-1}+\frac{1}{e^z-1}\right)\\ &= \frac{2}{z^2}\left(\frac{z^2e^z}{2e^z-2}+\frac{z^2}{2e^z-2}\right) = \frac{2}{z^2}\left(\sum_{n=0}^{\infty}R_n(1)\frac{z^n}{n!}+\sum_{n=0}^{\infty}R_n(0)\frac{z^n}{n!}\right).\end{aligned} \tag{26}$$

As $R_n(1)=R_n(0)=R_n \quad \forall n>2$; $R_2(1)=\frac{1}{2}$, $\quad R_2(0)=-\frac{1}{2}$, $\quad R_0(0)=0=R_0(1)$, $\quad R_1(1)=\frac{1}{2}=R_1(0)$.
Then (26) is transformed into

$$\coth\left(\frac{z}{2}\right) = \frac{2}{z^2}\left(z+2\sum_{n=3}^{\infty}\frac{R_n}{n!}z^n\right),$$

and given $R_n=0 \quad$ for n even with $n>3$ we have,

$$\frac{2}{z^2}\left(z+2\sum_{n=1}^{\infty}\frac{R_{2n+1}}{(2n+1)!}z^{2n+1}\right) = \frac{2}{z^2}\sum_{n=0}^{\infty}\frac{2R_{2n+1}}{(2n+1)!}z^{2n+1} = \sum_{n=0}^{\infty}4\frac{R_{2n+1}}{(2n+1)!}z^{2n-2}.$$

So

$$\coth\left(\frac{z}{2}\right) = \sum_{n=0}^{\infty}4\frac{R_{2n+1}}{(2n+1)!}z^{2n-2}.$$

Thus,

$$\coth(\pi iz) = \sum_{n=0}^{\infty}4\frac{R_{2n+1}}{(2n+1)!}(2\pi iz)^{2n-2}. \tag{27}$$

From (22), (25) and (27), we get

$$\begin{aligned} g(z) &= \pi\frac{\cot(\pi z)}{z^k}\\ &= \frac{\pi}{z^k}i\coth(\pi iz)\\ &= \frac{\pi}{z^k}i\sum_{n=0}^{\infty}4\frac{R_{2n+1}}{(2n+1)!}(2\pi iz)^{2n-1}\\ &= \sum_{n=0}^{\infty}(-1)^n 2(2\pi)^{2n}\frac{R_{2n+1}}{(2n+1)!}(z)^{2n-k-1}.\end{aligned}$$

Taking Laurentś series development of $g(z)$. It is enough to find the coefficient a_{-1}, that equals to $Res_{z=0}g(z)$.
So

$$Res_{z=0}g(z) = (-1)^{\frac{k}{2}}2(2\pi)^k\frac{R_{k+1}}{(k+1)!}.$$

And finally replacing this result in (24), we obtain

$$\begin{aligned}\zeta(k) &= \sum_{n=1}^{\infty}\frac{1}{n^k}\\ &= -\frac{1}{2}(-1)^{\frac{k}{2}}2(2\pi)^k\frac{R_{k+1}}{(k+1)!}\\ &= (-1)^{\frac{k}{2}+1}(2\pi)^k\frac{R_{k+1}}{(k+1)!}.\end{aligned}$$

□

Example 4.2. Some values of ζ are given by

$$\begin{aligned}\zeta(2) &= 6(-1)^2(2\pi)^2\frac{R_3}{3}\\ &= 4\pi^2\frac{1/4}{3!}\\ &= \frac{\pi^2}{6}.\end{aligned}$$

$$\begin{aligned}\zeta(4) &= (-1)^3(2\pi)^4\frac{R_5}{5!}\\ &= \frac{-16\pi^4(-1/12)}{120}\\ &= \frac{\pi^4}{90}.\end{aligned}$$

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