Dolomites Research Notes on Approximation

Volume 11 · 2018 · Pages 29–36

On the approximation of rapidly oscillatory Hankel transform via radial kernels

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Communicated by Alessandra De Rossi

Abstract

In this paper a kernel based method for the approximation of finite highly oscillatory Hankel transform is proposed. Such types of integrals arise in electromagnetic and acoustic scattering problems. A numerical scheme is constructed which is based on the local kernel based method [13]. The problem of evaluating such type of integrals is converted to a system of ODEs having non oscillatory solution [8] and the resultant ODEs are solved using radial kernels. The proposed method has the capability to get better accuracy for high frequency parameter σ. The numerical experiments are performed to illustrate accuracy and efficiency of the present method.

1 Introduction

Approximating the integrals of functions containing large oscillatory parameters is an important issue in the field of computational sciences. The computation of oscillatory integrals has many applications for example in acoustic scattering, electro-magnetics and engineering sciences. One such problem is time-harmonic acoustic plane wave T^i by a sound soft bounded convex obstacle D. In such problem the total acoustic field $T = T^i + T^s$ satisfies the differential equation of the form (see for example [17])

$$\nabla^2 T(x) + \sigma^2 T(x) = 0, \text{ where } x \in \Omega = R^d \setminus \bar{D}, \tag{1}$$

$$T(x) = 0, \text{ where } x \in \partial D, \tag{2}$$

and $\Omega \subset \mathbb{R}^d$, $\bar{D}$ is the closure of D, $d = 2, 3$, T^s is a scattered field, and wave number σ is proportional to frequency of corresponding incident wave. Let $\Psi(x, y)$ denote the fundamental solution of Helmholtz equation, then the $2D$ solution is given by

$$\Psi(x, y) = \frac{1}{4} H_0^{(1)}(\sigma|x - y|),$$

while the $3D$ solution is

$$\Psi(x, y) = \frac{\exp(i\sigma|x - y|)}{4\pi|x - y|},$$

where $y,\ x\ \in \mathbb{R}^d$, $y \neq x$, and

$$H_\nu^{(1)} = J_\nu(x) + iY_\nu(x),$$

is the first kind Hankel function of order ν. The solution $T(x)$ may be represented as (see for example [1, 2, 3])

$$T(x) = T^i(x) - \int_\Gamma \Psi(x, y)\nu(y)ds(y),\ x \in \Omega. \tag{3}$$

In the process of finding such type of solutions we always need for the case $d = 2$ and $d = 3$, to compute integrals of the form

$$I_1 = \int_a^b f(x) H_0^{(1)}(\sigma g(x))dx,\ \ I_2 = \int_a^b \int_c^d f(x, y) e^{i\sigma g(x,y)} dxdy. \tag{4}$$

The computation of integral of type $\int_a^b f(x)e^{i\omega g(x)}dx$ has the role to construct the quadrature rule for integral I_2. The integral $\int_a^b f(x)e^{i\omega g(x)}dx$ has extensively been studied. Some of the most efficient methods are available for computing such types of oscillatory integrals (see for example [5, 6, 9, 10]). Most recently some other methods (see for eaxmple [7, 11, 15, 16, 13]) have been developed for computing such types of integrals. The methods mentioned above for computing the integral $\int_a^b f(x)e^{i\omega g(x)}dx$

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may not be applicable to integral I_1 directly, but the composite technique can be used very effectively. In the present work we extended the work [8] to construct a radial kernel based numerical scheme for evaluating integrals of the type

$$I = \int_a^b f(x) H_\nu^{(1)}(\sigma x) dx, \tag{5}$$

where σ is a large oscillation parameter and $\nu \in [0,1]$. In the present procedure the problem of integral computation is converted into a system of ODEs without any boundary condition satisfying some differential condition. The resultant system of ODEs is approximated with localized kernel based method [13]. The present extended formulation to the highly oscillatory finite Hankel transform works well for very large oscillation parameter σ.

2 Preliminaries

Theorem 2.1. *[4] Assume the expansion of positive definite kernel function κ in the form*

$$\kappa(r) = \sum_{n=0}^{\infty} a_n r^{2n}$$

where κ is radial and infinitely smooth, then for N data locations X,

$$lim_{\varepsilon \to 0} \upsilon_\varepsilon(x) = p_m(x), \ x \in \mathbb{R}^d,$$

and the polynomial interpolant p_m depends on the choice of specific radial kernel function.

Theorem 2.1 holds for the kernels shown in Table 1. This implies that radial kernel based methods are generalization of spectral methods.

Table 1: Radial kernels satisfy the conditions of Theorem 2.1

IQ	$\frac{1}{1+r^2}$	$1 - r^2 + r^4 - r^6 + \ldots$
Gaussian	e^{-r^2}	$1 - r^2 + \frac{r^4}{2} - \frac{r^6}{6} + \ldots$
IMQ	$\frac{1}{\sqrt{1+r^2}}$	$1 - \frac{r^2}{2} + \frac{3r^4}{8} - \frac{5r^6}{16} + \ldots$
Poisson	$J_0(r)$	$1 - \frac{r^2}{4} + \frac{r^4}{64} - \frac{r^6}{2304} + \ldots$

Lemma 2.2. *[4] Let $\{x_i, f_i\}_{i=1}^N$ be N data points in $X = \{x_1, \ldots, x_N\} \subset \Omega = [a,b]$, let $h = sup_{x \in \Omega} min_{x_j \in X} \|x - x_j\|$ be the fill distance in Ω, the kernel-based interpolant $s \in C^\beta(\Omega)$ for function $f \in C^\beta(\Omega)$ for all points in X, then the error estimate*

$$\| f - s \|_{L_2(\Omega)} \leq 2Ch^\beta \mid f \mid_{W_2^\beta},$$

where the W_2^β is the Sobolev space, $\beta \leq N$, and $C = max\{C_1\sqrt{b-a}, C_2\}$, where C_1 and C_2 depend on β and N. Further if $l \in \mathbb{N}$ and $\mid \alpha \mid \leq l$, then

$$\mid D^\alpha f(x) - D^\alpha s(x) \mid \leq C_l h^{l-|\alpha|} \mid f \mid_{W_2^\beta}.$$

Lemma 2.3. *[8] Given a vector $\mathbf{u}(x) = (u_1(x), u_2(x), \ldots, u_m(x))^t$ which satisfies (9) and $q(x)$ be monotonic over $[a,b]$, then $\mathbf{w}(x) = \mathbf{u}(q(x))$ satisfying the equation*

$$\mathbf{w}'(x) = \mathbf{B}(x)\mathbf{w}(x), \tag{6}$$

with $\mathbf{B}(x)$ is of order $m \times m$ containing functions which are non-rapid oscillatory.

Lemma 2.4. *[8] Suppose the vectors $\mathbf{w} = (w_1(x), w_2(x), \ldots, w_k(x))^t$ and $\mathbf{z} = (z_1(x), z_2(x), \ldots, z_l(x))^t$ satisfy the following equations*

$$\mathbf{w}'(x) = \mathbf{B}_1(x)\mathbf{w}(x), \tag{7}$$

and

$$\mathbf{z}'(x) = \mathbf{B}_2(x)\mathbf{z}(x), \tag{8}$$

respectively, where matrices $\mathbf{B}_1(x)$ and $\mathbf{B}_2(x)$ are matrices of order $k \times k$, and $l \times l$ of non-highly oscillatory functions. Then the vector $\mathbf{u} = \{w_j z_i | j = 1, \ldots, k, i = 1, \ldots, l\}$ satisfies the equation

$$\mathbf{u}'(x) = \mathbf{A}(x)\mathbf{u}(x) \tag{9}$$

with $m = kl$ and $\mathbf{A}(x)$ matrix of order $m \times m$ of non-highly oscillatory functions.

3 Localized kernel based method

Consider a more generalized class of rapidly oscillatory integrals like

$$I = \int_a^b \mathbf{f}^t(x)\mathbf{u}(x)dx \equiv \int_a^b \langle \mathbf{f}, \mathbf{u}\rangle(x)dx, \tag{10}$$

where $\mathbf{f}^t(x)$ denote the transpose of $\mathbf{f}(x)$ given by

$$\mathbf{f}(x) = (f_i(x), i = 1, ..., m)^t$$

and is a vector of non-rapidly oscillatory functions, and

$$\mathbf{u}(x) = (u_i(x), i = 1, ..., m)^t$$

is a vector of linearly independent rapidly oscillatory functions. It is shown in the work [8] that $\{u_i\}_{i=1}^m$ satisfies the ODE system

$$\mathbf{u}'(x) = A(x)\mathbf{u}(x), \tag{11}$$

and eventually the matrix $A(x)$ of order $m \times m$ becomes a matrix of non-oscillatory functions. The work [8] leads to approximate I in (10) by the derivative of given known function. It is assumed to find

$$\mathbf{p}(x) = (p_1(x), ..., p_m(x))^t,$$

such that

$$\langle \mathbf{p}, \mathbf{u}\rangle' \approx \langle \mathbf{f}, \mathbf{u}\rangle. \tag{12}$$

Thus the integral in (10) is approximated by

$$I \approx \int_a^b \langle \mathbf{p}, \mathbf{u}\rangle'(x)dx = \mathbf{p}^t(b)\mathbf{u}(b) - \mathbf{p}^t(a)\mathbf{u}(a). \tag{13}$$

Expanding (12) and using (11) we get

$$\langle \mathbf{p}, \mathbf{u}\rangle' = \langle \mathbf{p}', \mathbf{u}\rangle + \langle \mathbf{p}, \mathbf{u}'\rangle = \langle \mathbf{p}', \mathbf{u}\rangle + \langle \mathbf{p}, A\mathbf{u}\rangle = \langle \mathbf{p}' + A^t\mathbf{p}, \mathbf{u}\rangle \approx \langle \mathbf{f}, \mathbf{u}\rangle. \tag{14}$$

By the assumption that $\{u_i\}_{i=1}^m$ are linearly independent it implies that $\mathbf{p}$ must approximate solution of the ODEs system

$$\mathcal{L}\mathbf{p} = \mathbf{p}' + A^t\mathbf{p} = \mathbf{f}, \tag{15}$$

where the function $\mathbf{f}$ and the matrix A are non rapidly oscillatory. It was investigated in the work [10], that the system (15) may have a solution which is not oscillatory at all. This non-oscillatory solution of the PDEs system can be approximated accurately by collocation methods using some suitable basis functions. In the present work we extended the idea [8] to approximate the linear differential operator $\mathcal{L}$ and construct a sparse differentiation matrix corresponding to (15). To construct local interpolant (see for example [12, 13, 14]), at each center $x_i \in \Omega_i \subset \Omega$, we define

$$v^i(x_k) = \sum_{x_j \in \Omega_i} c_j^i \phi^i(\|x_k - x_j\|),\ x_k, x_j \in \Omega_i, \tag{16}$$

where c_j are the expansion coefficients, $\|x_k - x_j\|$, denotes the norm of the difference of centers x_k and x_j, $\phi(r)$ a radial kernel, with the radial distance $r \geq 0$, and $\Omega_i \subset \Omega$ is a local sub-domain corresponding to each center x_i and contains n nearest centers around the center x_i. For each node x_i, we get the $n \times n$ linear systems

$$\mathbf{v}^i = \mathbf{B}^i\mathbf{c}^i, \quad i = 1, ..., N, \tag{17}$$

where the matrix $\mathbf{B}^i$ has the elements $b_{kj}^i = \phi(\|x_k - x_j\|), x_k, x_j \in \Omega_i$.
Next we apply the operator $\mathcal{L}$ to (16) and get

$$\mathcal{L}v^i(x_i) = \sum_{x_j \in \Omega_i} c_j^i \mathcal{L}\phi^i(\|x_i - x_j\|). \tag{18}$$

The expression in (18) may be given by dot product of two vectors,

$$\mathcal{L}\mathbf{v}^i(x_i) = \mathbf{w}^i \cdot \mathbf{c}^i, \tag{19}$$

where the entries of the vector $\mathbf{w}^i$ are given by

$$\mathcal{L}\phi^i(\|x_i - x_j\|), \quad x_j \in \Omega_i. \tag{20}$$

Eliminating the coefficients $\mathbf{c}^i$ from (17) and (19) we get

$$\mathcal{L}\mathbf{v}^i(x_i) = \mathbf{w}^i(\mathbf{B}^i)^{-1}\mathbf{v}^i = \mathbf{D}^i\mathbf{v}^i, \tag{21}$$

where

$$\mathbf{D}^i = \mathbf{w}^i(\mathbf{B}^i)^{-1} \tag{22}$$

is a row vector of order $1 \times N$ containing n non-zero entries and remaining $N - n$ zeros entries. Consequently for all centers $x_i, i = 1, 2, 3, ..., N$, the differential operator $\mathcal{L}$ can be approximated by a sparse differentiation matrix $\mathbf{D}$ of order $N \times N$ given by

$$\mathcal{L}\mathbf{v} = \mathbf{D}\mathbf{v}. \tag{23}$$

4 Global kernel based method

For the given points $x_1, x_2, ..., x_N \in \Omega$. The function $p(x)$ may be approximated as linear combination of radial kernels $\phi(r)$,

$$p(x) = \sum_{j=1}^{N} c_j \phi(\|x - x_j\|), \quad x_j \in \Omega, \quad j = 1, 2, 3, ..., N. \tag{24}$$

Let $\mathcal{L}$ be the linear operator then from (24), we get

$$\mathcal{L}p(x) = \sum_{j=1}^{N} c_j \mathcal{L}\phi(\|x - x_j\|), \quad x_j \in \Omega, \quad j = 1, 2, 3, ..., N. \tag{25}$$

To approximate the system of ODEs (15), we compute (25) for the evaluation points $x \in \{x_1, x_2, ..., x_N\} \subset \Omega$, and get

$$\mathbf{Dc} = \mathbf{f}, \tag{26}$$

where $\mathbf{c}$ is a $N \times 1$ vector of expansion coefficients and $\mathbf{D}$ is a $N \times N$ matrix with entries $\mathcal{L}\phi(\|x_i - x_j\|)_{i,j=1}^{N}$, and $\mathbf{f}$ is $N \times 1$ vector. The solution of (15) using the global kernel based method is given by

$$\mathbf{p} = \mathbf{Mc}, \tag{27}$$

where $\mathbf{M}$ is $M \times N$ evaluation matrix with the entries $\phi(\|x_i - x_j\|)_{i=1,...M, j=1,...N}$, and the value of $\mathbf{c}$ can be obtained from (26).

5 Application of the proposed method

This section is devoted to demonstrate the validity and applicability of the kernel based local method to the highly oscillatory Hankel transform.

5.1 Type 1: For integral value of ν.

We consider the integral of the type

$$I_n = \int_a^b f(x) H_\nu^{(1)}(\sigma x) dx. \tag{28}$$

The bases corresponding to this integral can be obtained from the recurrence relations of Hankel functions given by

$$\begin{bmatrix} H_{\nu-1}^{(1)}(\sigma x) \\ H_\nu^{(1)}(\sigma x) \end{bmatrix}' = \begin{bmatrix} (\nu-1)/x & -\sigma \\ \sigma & -\nu/x \end{bmatrix} \begin{bmatrix} H_{\nu-1}^{(1)}(\sigma x) \\ H_\nu^{(1)}(\sigma x) \end{bmatrix}. \tag{29}$$

It follows that the vector

$$u(x) = \left[H_{\nu-1}^{(1)})(\sigma x), H_\nu^{(1)}(\sigma x)\right]^t$$

satisfies the differential condition (11) with the corresponding matrix given by

$$A = \begin{bmatrix} (\nu-1)/x & -\sigma \\ \sigma & -\nu/x \end{bmatrix}. \tag{30}$$

So for $\nu = 1$, the kernel based method is applied to approximate the integrals like given below

$$\int_a^b (f_1(x) H_0^{(1)}(\sigma x) + f_2(x) H_1^{(1)}(\sigma x)) dx. \tag{31}$$

The local kernel based approximate scheme of the ODE system (15), corresponding to the integral (28), is given by

$$\begin{bmatrix} p_1' + ((\nu-1)/x) p_1 & \sigma p_2 \\ -\sigma p_1 & p_2' - ((\nu)/x) p_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{32}$$

For $\nu = 1$, the above system can be given by

$$\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \tag{33}$$

where

$$d_{11} = D_x, \; d_{12} = \sigma A_{rbf}, \; d_{21} = -\sigma A_{rbf}, \; d_{22} = D_x - \text{diag}\left(\frac{1}{x}\right) A_{rbf}.$$

Here the $N \times N$ matrix D_x can be obtained from (23). The matrix A_{rbf} is also $N \times N$ and can be obtained from (23) and $f_1(x) \equiv f(x)$, $f_2(x) \equiv 0$. So approximation to the integral in (28) is given by the following numerical scheme

$$I_n = \sum_{i=1}^{2} [u_i(b) p_i(b) - u_i(a) p_i(a)]. \tag{34}$$

Table 2: Absolute errors and computations times, corresponding to integral (35) using the proposed numerical scheme.

	LK-method			GK-method
σ	$(n,N)=(3,5)$	$(n,N)=(3,10)$	$(n,N)=(3,5000)$	$N=5$
1	2.20e-03	1.00e-03	5.51e-08	6.43e-04
10	2.46e-05	9.43e-05	1.17e-07	3.15e-05
100	4.78e-08	8.58e-09	2.22e-09	1.70e-08
1000	6.04e-10	3.46e-10	3.01e-10	2.93e-10
σ	$(n,N)=(5,5)$	$(n,N)=(5,10)$	$(n,N)=(5,5000)$	$N=10$
1	1.1264e-004	2.40e-03	2.74e-06	3.32e-06
10	2.7584e-005	8.41e-05	3.25e-07	4.40e-08
100	8.6895e-008	1.06e-08	9.91e-09	1.96e-10
1000	5.0247e-010	1.38e-09	1.17e-08	2.39e-12

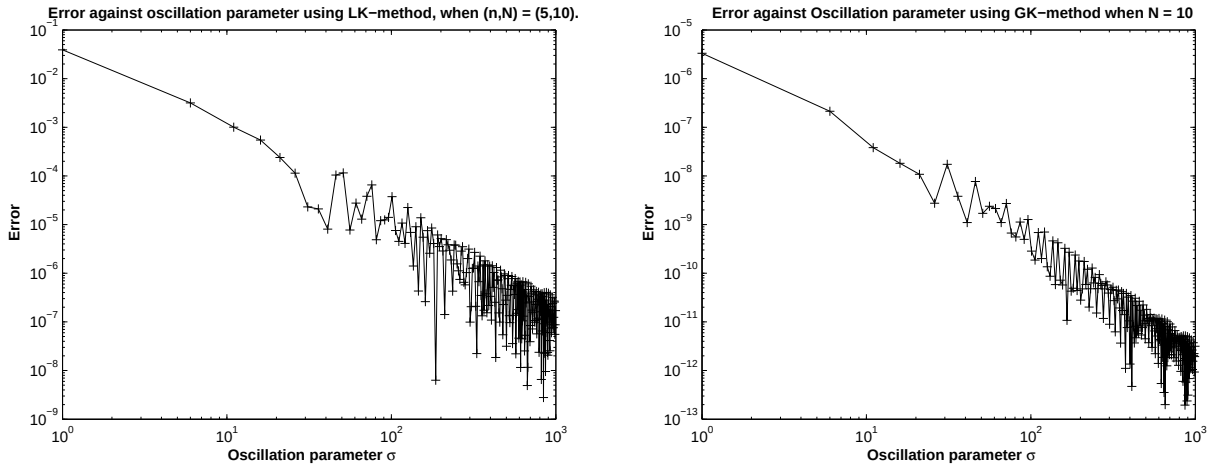

Figure 1: These plots show the error against oscillation parameter σ obtained with LK-method and GK-method respectively, corresponding to Integral (35).

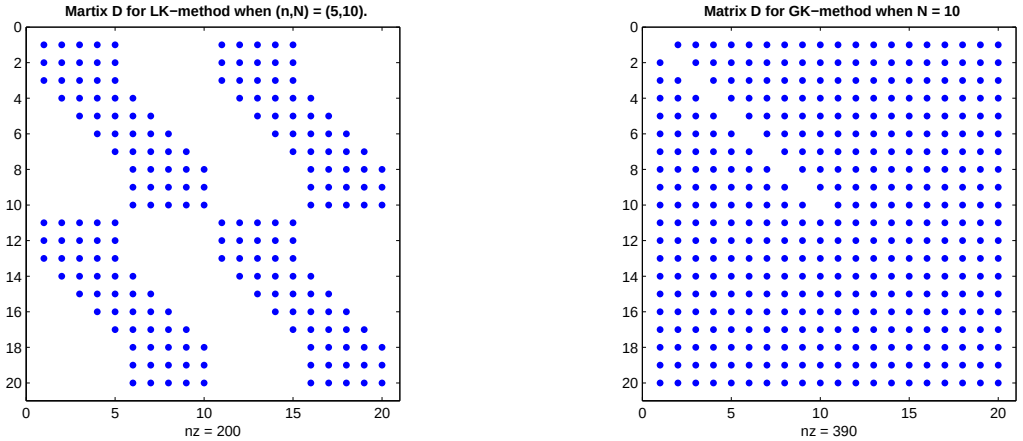

Figure 2: These plots show the sparsity of the collocation matrices obtained with LK-method and GK-method respectively, corresponding to Integral (35).

The results of the present method corresponding to the integral

$$\int_1^2 (x^2+1)^{-1} H_0^{(1)}(\sigma x)dx, \tag{35}$$

are shown in Table 2. The results show very fast convergence rate even for small number of collocation points and large oscillation parameter σ.

We approximated the integral (35) using the LK-method discussed in section 3 and GK-method in section 4 respectively. The results are shown in Table 2, and Figures 1-2. We used different number of nodes n in local domain $\Omega_i \subset \Omega$ and N in global domain Ω. The advantage of the local method over the global method is that the resultant differentiation matrix is sparse while that obtained with the global method is dense. The $N \times N$ sparse matrix is assembled by solving small size matrices of order $n \times n$ in each local sub-domain $\Omega_i, i = 1, 2, ..., N$, where $n \ll N$. The LK-method can be used for large number of collocations points e.g $N = 5000$ or more, while the GK-method can not be used for such a large number of points due to dense differentiation matrix, as shown in Table 2. The LK-method $\rightarrow$ GK-method when $n \rightarrow N$.

5.2 Type 2: For non-integral value of ν.

We consider integral of the type

$$I_n = \int_a^b f(x) H_\nu^{(1)}(\sigma x)dx. \tag{36}$$

The required basis for this integral are the Hankel functions and can be obtained from the recurrence relations of the Hankel functions

$$\begin{bmatrix} H^{(1)}_{\eta-\frac{1}{2}}(\sigma x) \\ H^{(1)}_{\eta+\frac{1}{2}}(\sigma x) \end{bmatrix}' = \begin{bmatrix} (\eta-\frac{1}{2})/x & -\sigma \\ \sigma & (\frac{1}{2}-\eta)/x \end{bmatrix} \begin{bmatrix} H^{(1)}_{\eta-\frac{1}{2}}(\sigma x) \\ H^{(1)}_{\eta+\frac{1}{2}}(\sigma x) \end{bmatrix}. \tag{37}$$

The vector of basis functions

$$u(x) = \left[H^{(1)}_{\eta-\frac{1}{2}}(\sigma x), H^{(1)}_{\eta+\frac{1}{2}}(\sigma x) \right],$$

satisfies the differential condition (11) with the corresponding matrix given by

$$A = \begin{bmatrix} (\eta-\frac{1}{2})/x & -\sigma \\ \sigma & (\frac{1}{2}-\eta)/x \end{bmatrix}. \tag{38}$$

The proposed method is applied to approximate the integral given by

$$\int_a^b (f_1(x) H^{(1)}_{1/2}(\sigma x) + f_2(x) H^{(1)}_{3/2}(\sigma x))dx. \tag{39}$$

The numerical scheme of the ODE system (15), corresponding to the integral (36)

$$\begin{bmatrix} p_1' + ((\eta-1/2)/x)p_1 & \sigma p_2 \\ -\sigma p_1 & p_2' - ((\eta-1/2)/x)p_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{40}$$

For $\eta = 1$ the above system is given by

$$\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \tag{41}$$

where

$$d_{11} = D_x + \text{diag}\left(\frac{1}{2x}\right) A_{rbf},\ d_{12} = \sigma A_{rbf},\ d_{21} = -\sigma A_{rbf},\ d_{22} = D_x - \text{diag}\left(\frac{1}{2x}\right) A_{rbf}.$$

Here the $N \times N$ matrix D_x can be obtained from (23). The matrix A_{rbf} is also $N \times N$ and can be obtained from (23) and $f_1(x) \equiv f(x),\ f_2(x)) \equiv 0$.

And consequently integral in (36) is given by the following numerical scheme

$$I_n = \sum_{i=1}^{2} [u_i(b)P_i(b) - u_i(a)P_i(a)]. \tag{42}$$

The results of the present method corresponding to the integral

$$\int_0^1 \frac{e^x}{1+100(x-1/2)^2} H^{(1)}_{1/2}(\sigma x)dx, \tag{43}$$

are shown in Table 3. The present numerical scheme performed very well for non-integral value of ν and for large oscillation parameter σ.

Table 3: Approximate value of the integral (43) using the kernel based numerical scheme.

	LK-method		GK-method	
σ	$(n,N)=(7,10)$	$(n,N)=(7,20)$	$N=5$	$N=20$
1	1.41e-02	1.02e-02	7.10e-03	1.09e-02
10	4.75e-02	2.57e-02	7.29e-02	1.76e-02
100	2.55e-05	2.67e-05	3.36e-04	4.19e-05
1000	2.51e-08	2.97e-08	5.22e-08	7.27e-08

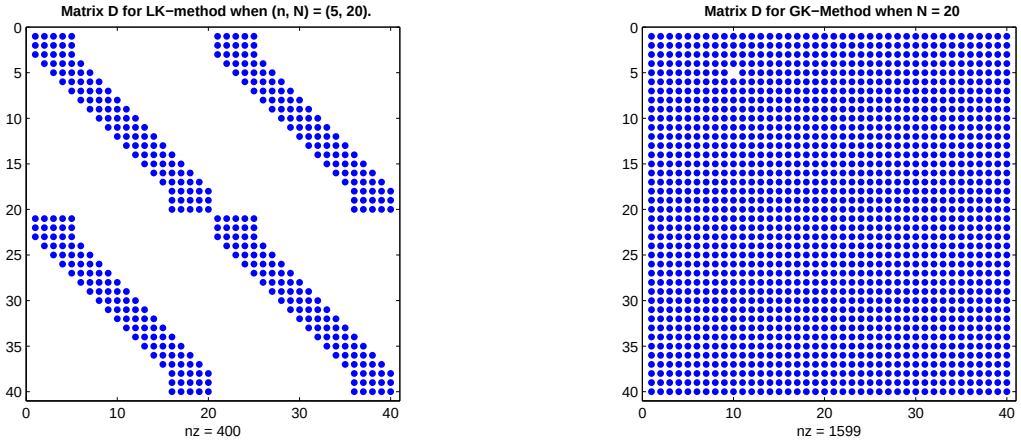

Figure 3: These plots show the sparsity of the collocation matrices obtained with LK-method and GK-method respectively, corresponding to Integral (43).

6 Conclusion

The proposed kernel based numerical scheme which is based on the work [8] in the context of radial kernel functions for approximating the highly oscillatory finite Hankel transform is accurate. Although the accuracy of GK-method method is better than LK-method, yet LK-method can be applied over irregular domain Ω for large number of points. The use of radial kernels in the local setting e.g. [12] is much suited for computing the finite Hankel transform. In the present collocation method the differentiation matrix is sparse contrary to dense ill-conditioned differentiation matrix in the global method, and can be solved with much ease due to small size system matrices. Thousands of scattered collocations points can be incorporated. The real benefit of the present method is its applicability for computing such types of highly oscillatory finite transforms.

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