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On polynomial and barycentric interpolations

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Abstract

The present survey collects some recent results on barycentric interpolation showing the similarity to the corresponding Lagrange (or polynomial) theorems. Namely we state that the order of the Lebesgue constant for barycentric interpolation is at least $\log n$; we state a Grünwald–Marcinkiewicz type theorem for the barycentric case; moreover we define a Bernstein type process for the barycentric interpolation which is *convergent for any continuous function*. As far as we know this is *the first process of this type*. The analogue results for the polynomial Lagrange interpolation are well known.

1 Introduction

In the last 30 years the barycentric interpolation has been investigated first of all considering its practical importance. The aim of this paper is to state some recent results showing the similarity between the barycentric and polynomial Lagrange interpolation. Among others we see that the order of the corresponding Lebesgue constant is at least $\log n$. We state a Grünwald–Marcinkiewicz type theorem for the barycentric case. Moreover we define the analogue of the polynomial Bernstein process for the barycentric interpolation. As far as we know it is the first barycentric interpolation procedure which is convergent *for every continuous function*. The analogon for the polynomial case is well known.

2 Lagrange and barycentric interpolation

2.1

Let $C = C(I)$ denote the space of continuous functions on the interval $I := [-1, 1]$, and let $\mathcal{P}_n$ denote the set of algebraic polynomials of degree at most n. $\|\cdot\|$ stands for the usual maximum norm on C. Let X be an *interpolatory matrix (array)*, i.e.,

$$X = \{x_{kn} = \cos\vartheta_{kn}; \qquad k = 1, \dots, n; \qquad n = 1, 2, \dots\},$$

with

$$-1 = x_{n+1,n} \le x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n} \le x_{0n} = 1 \tag{2.1}$$

and $0 \le \vartheta_{kn} \le \pi$. Consider the corresponding *Lagrange interpolation polynomial*

$$L_n(f, X, x) := \sum_{k=1}^{n} f(x_{kn})\ell_{kn}(X, x), \qquad n \in \mathbb{N}. \tag{2.2}$$

Here, for $n \in \mathbb{N}$,

$$\ell_{kn}(X, x) := \frac{\omega_n(X, x)}{\omega_n'(X, x_{kn})(x - x_{kn})}, \qquad 1 \le k \le n,$$

with

$$\omega_n(X, x) := \prod_{k=1}^{n}(x - x_{kn}),$$

are polynomials of exact degree $n-1$. They are the *fundamental polynomials*, obeying the relations $\ell_{kn}(X, x_{jn}) = \delta_{kj}$, $1 \le k, j \le n$.

By the classical Lebesgue estimate, using the notations

$$\lambda_n(X, x) := \sum_{k=1}^{n} \left|\ell_{kn}(X, x)\right|, \qquad n \in \mathbb{N}, \tag{2.3}$$

$$\Lambda_n(X) := \left\|\lambda_n(X, x)\right\|, \qquad n \in \mathbb{N}, \tag{2.4}$$

(*Lebesgue function* and *Lebesgue constant* (of Lagrange interpolation), respectively,) we obtain if $n \in \mathbb{N}$

$$\left|L_n(f, X, x) - f(x)\right| \le \left\{\lambda_n(X, x) + 1\right\}E_{n-1}(f) \tag{2.5}$$

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and

$$\left\|L_n(f,X)-f\right\| \le \left\{\Lambda_n(X)+1\right\}E_{n-1}(f), \tag{2.6}$$

where

$$E_{n-1}(f) := \min_{P\in\mathcal{P}_{n-1}} \|f-P\|.$$

The above estimates clearly show the importance of $\lambda_n(X,x)$ and $\Lambda_n(X)$.

2.2

Using the obvious identity

$$\sum_{k=1}^{n} \ell_{kn}(X,x) \equiv 1, \tag{2.7}$$

we can write $L_n(f,X,x)$ as

$$L_n(f,X,x) = \frac{\omega_n(X,x)\sum_{k=1}^{n} \frac{w_{kn}}{x-x_{kn}} f(x_{kn})}{\omega_n(X,x)\sum_{k=1}^{n} \frac{w_{kn}}{x-x_{kn}}}, \tag{2.8}$$

where

$$w_{kn} = \frac{1}{\omega_n'(X,x_{kn})} = \frac{1}{\prod_{j\neq k}(x_{kn}-x_{jn})}, \qquad 1\le k\le n. \tag{2.9}$$

A simple consideration shows that the right-hand expression of (2.8) at x_{jn} takes the value $f(x_{jn})$, $1\le j\le n$, using *arbitrary* $w_{kn}\neq 0$, $1\le k\le n$. Thus, choosing $w_{kn}=(-1)^{k+1}$, we get the *classical barycentric interpolation* formula for $f\in C$:

$$B_n(f,X,x) := \sum_{k=1}^{n} f(x_{kn}) b_{kn}(X,x), \qquad n\in\mathbb{N}, \tag{2.10}$$

where

$$b_{kn}(X,x) := \frac{\omega_n(X,x)\frac{(-1)^k}{x-x_{kn}}}{\omega_n(X,x)\sum_{j=1}^{n}\frac{(-1)^j}{x-x_{jn}}} = \frac{\frac{(-1)^k}{x-x_{kn}}}{\sum_{j=1}^{n}\frac{(-1)^j}{x-x_{jn}}}. \tag{2.11}$$

The first equation of (2.11) shows that b_{kn} is a *rational function* of the form P_{kn}/Q_n, where

$$P_{kn}(X,x) = \left|\omega_n'(X,x_{kn})\right| \ell_{kn}(X,x), \qquad 1\le k\le n, \tag{2.12}$$

$$Q_n(X,x) = \sum_{j=1}^{n} \left|\omega_n'(X,x_{jn})\right| \ell_{jn}(X,x), \qquad n\in\mathbb{N}. \tag{2.13}$$

Above, $P_{kn}\in\mathcal{P}_{n-1}\setminus\mathcal{P}_{n-2}$ and $Q_n\in\mathcal{P}_{n-1}$.

As we remarked, the process B_n has the interpolatory property, i.e.

$$B_n(f,X,x_{kn}) = f(x_{kn}), \quad b_{kn}(X,x_{jn}) = \delta_{kj}, \qquad 1\le k,j\le n;\ n\in\mathbb{N} \tag{2.14}$$

(cf. (2.10) and (2.11)). Moreover, it is not so difficult to prove the next fundamental relation valid for *arbitrary* matrix X:

$$Q_n(X,x)\neq 0 \quad \text{if} \quad x\in\mathbb{R},\ \ n\in\mathbb{N}, \tag{2.15}$$

where $\mathbb{R}=(-\infty,\infty)$ (see J.-P. Berrut [3, Lemma 2.1]).

By the above definitions we can prove that $\{b_{kn}(x),\ 1\le k\le n\}$ is a Haar (Tchebycheff) system (or briefly, T-system) for any fixed $n\in\mathbb{N}$ (see [6] and [17]). Actually, $T=T(\mathbf{x}_n)$ where $\mathbf{x}_n=(x_{1n},x_{2n},\dots,x_{nn})\in\mathbb{R}^n$.

Modifying our previous notation (2.10), we define for $f\in C$, $\mathbf{x}_n$ and $n\in\mathbb{N}$

$$L_n(f,\mathbf{x}_n,x) := \sum_{k=1}^{n} f(x_{kn}) b_{kn}(\mathbf{x}_n,x), \tag{2.16}$$

$$\lambda_n(\mathbf{x}_n,x) := \sum_{k=1}^{n} \left|b_{kn}(\mathbf{x}_n,x)\right|, \tag{2.17}$$

$$\Lambda_n(\mathbf{x}_n) := \left\|\lambda_n(\mathbf{x}_n,x)\right\|, \tag{2.18}$$

(cf. (2.2)–(2.4)).

By definition, they are the Lagrange interpolatory *T-polynomials, T-Lebesgue functions and T-Lebesgue constants, respectively*, concerning the above defined T-system.

3 Divergence-type results

3.1

As we mentioned the estimation (2.6) shows the importance of $\lambda_n(X,x)$ and $\Lambda_n(X)$. So it was fundamental the result of G. Faber [7] from 1914 which says that

$$\Lambda_n(X) \geq \frac{1}{12} \log n, \qquad n \geq 1 \tag{3.1}$$

for any interpolatory X.

However we can prove much more. Namely improving some results of P. Erdős and P. Vértesi (cf. [16, 2.2.]) P. Vértesi [15] proved

Theorem 3.1. *There exists a positive constant c such that if $\varepsilon = \{\varepsilon_n\}$ is any sequence of positive numbers then for arbitrary matrix X there exist sets $H_n = H_n(\varepsilon, X)$, $|H_n| \leq \varepsilon_n$ for which*

$$\lambda_n(X,x) > c\varepsilon_n \log n$$

if $x \in [-1,1] \setminus H_n$ and $n = 1,2,\dots$.

Using (3.1) one can see that for any fixed matrix X there exists an $f \in C$ such that

$$\lim_{n\to\infty} \|L_n(f,X,x)\| = \infty.$$

3.2

Many papers deal with the barycentric interpolation (cf. J.-P. Berrut and G. Klein [4] and its references) and define pointsystem having T-Lebesgue constant of order $\log n$. We mention two recent papers. Firstly the work L. Bos, S. De Marchi, K. Hormann and J. Sidon [5], where the so called *well spaced nodes* are defined; secondly the paper of B. A. Ibrahimoglu and A. Cuyt [10] stating that for the nodes $\mathbf{e}_n = \left\{ e_{kn} = -1 + \frac{2k-1}{n} \; ; \; k = 1,2,\dots,n \right\}$

$$\Lambda_n(\mathbf{e}_n) = \frac{2}{\pi} \log n + O(1).$$

However, the next fundamental Faber-type statement, as far as we know, is new (cf. G. Halász [8]).

Theorem 3.2. *For arbitrary system $\mathbf{x}_n$*

$$\Lambda_n(\mathbf{x}_n) > \frac{\log n}{8}, \qquad n \geq 3. \tag{3.2}$$

More detailed considerations show that a statement analogous to Theorem 3.1 can be proved. Namely we have

Theorem 3.3. *There exists a positive constant c such that if $\varepsilon = \{\varepsilon_n\}$ is any sequence of positive numbers, then for arbitrary matrix X there exist sets $H_n = H_n(\varepsilon, X)$, $|H_n| \leq \varepsilon_n$, for which*

$$\lambda_n(\mathbf{x}_n, x) > c\varepsilon_n \log n \tag{3.3}$$

if $x \in [-1,1] \setminus H_n$ and $n = 1,2,\dots$.

The proofs of these are in [17].

3.3

A fundamental negative result in Lagrange interpolation is due to G. Grünwald and J. Marcinkiewicz from 1936.

Theorem 3.4. *There exists a function $f \in C$ for which*

$$\overline{\lim_{n\to\infty}} \left| L_n(f,T,x) \right| = \infty$$

for every $x \in [-1,1]$, where $T = \left\{ t_{kn} = \cos \frac{2k-1}{2n}\pi \; ; \; k = 1,2,\dots,n; \; n \in \mathbb{N} \right\}$.

The reader can consult for the interesting history of Theorem 3.4 in [16, p. 76].

Now we quote the corresponding result for the *barycentric case* proved by Ágota P. Horváth and P. Vértesi, in 2015 (see [9]).

Theorem 3.5. *One can define a $g \in C$ such that*

$$\overline{\lim_{n\to\infty}} \left| L_n(g, \mathbf{e}_n, x) \right| = \infty$$

for every $x \in [-1,1]$.

Let us remark that in both cases

$$\Lambda_n(T) = \frac{2}{\pi} \log n + O(1) \qquad \text{and} \qquad \Lambda_n(\mathbf{e}_n) = \frac{2}{\pi} \log n + O(1),$$

so they have the best possible order!

4 On the convergence of a Bernstein type process

4.1

Let be the *polynomials* $Q_{n1}(f,T,x)$ defined according to S. Bernstein [1] and [2]

$$Q_{n1}(f,T,x)=\sum_{k=1}^{n/2} f(t_{2k-1,n})\left\{\ell_{2k-1,n}(T,x)+\ell_{2k,n}(T,x)\right\}, \qquad f\in C \tag{4.1}$$

(for simplicity let n be even). Actually, [1] and [2] took the next more general process. Let l,q be natural numbers; for simplicity we suppose that $n=2lq$. We divide the nodes into q rows as follows.

$$\begin{array}{cccc} x_{1n} & x_{2n} & \dots & x_{2l,n} \\ x_{2l+1,n} & x_{2l+2,n} & \dots & x_{4l,n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{2l(q-1)+1,n} & x_{2l(q-1)+2,n} & \dots & x_{2lq,n}. \end{array}$$

According to the above scheme let

$$\begin{aligned} Q_{nl}(f,T,x)&\equiv Q_{nl}(f)= \\ &=\left\{f_1(\ell_1+\ell_{2l})+f_2(\ell_2-\ell_{2l})+f_3(\ell_3+\ell_{2l})+\dots+f_{2l-1}(\ell_{2l-1}+\ell_{2l})\right\}_1+ \\ &+\left\{f_{2l+1}(\ell_{2l+1}+\ell_{4l})+f_{2l+2}(\ell_{2l+2}-\ell_{4l})+f_{2l+3}(\ell_{2l+3}+\ell_{4l})+\dots+f_{4l-1}(\ell_{4l-1}+\ell_{4l})\right\}_2+\dots+ \\ &+\left\{f_{n-(2l-1)}(\ell_{n-(2l-1)}+\ell_n)+\dots+f_{n-1}(\ell_{n-1}+\ell_n)\right\}_q. \end{aligned} \tag{4.2}$$

You may consult with [1] or [2] (above $f_k=f(x_{kn})$ and $\ell_k\equiv\ell_{kn}(T,x)$).

If $N=n+r$, $n=2lq$, $0<r<2l$, the definition of Q_{Nl} is as follows

$$Q_{Nl}(f):=Q_{nl}(f)+\sum_{k=n+1}^{N} f_k\ell_k.$$

By the above definitions we have with $e_0(x)\equiv 1$

$$Q_{nl}(e_0,x)\equiv\sum_{k=1}^{n}\ell_{kn}(T,x)\equiv 1, \tag{4.3}$$

$$Q_{nl}(f,x_{kn})=f(x_{kn}) \quad \text{if} \quad k\neq 2l,\,4l,\,\dots,\,2lq, \tag{4.4}$$

i.e. Q_{nl} interpolates at $n-q=2lq-q$ nodes. This number is "very close" to n if the (fixed) l is big enough while q (and n, too) tends to infinity, i.e. for large l our Q_{nl} is "very close" to the Lagrange interpolation L_n. However, Q_{nl} converges for every $f\in C$, when $n\to\infty$ (cf. Theorem 4.1 and Theorem 4.2), which generally does not hold for L_n.

Actually, (4.3) and (4.4) hold true for *arbitrary* point system.

4.2

In [1] S. Bernstein proved

Theorem 4.1. *Let l be a fixed positive integer and $f\in C$. Then*

$$\lim_{n\to\infty}\|f(x)-Q_{nl}(f,x)\|=0.$$

Actually, he proved for $N=n+r$, too; the case when $N=n+r$ demands only small technical changes in the proof.

Remark 1. The Bernstein process and its generalizations were exhaustively investigated by O. Kis (sometimes with coauthors). For more details we suggest the references in [12].

4.3

In [12] we generalized Theorem 4.1 using the roots of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. So let $P_n^{(\alpha,\beta)}(x)$ be defined by

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\left\{(1-x)^{\alpha+n}(1+x)^{\beta+n}\right\} \qquad (\alpha,\beta>-1).$$

For the roots $x_{kn}^{(\alpha,\beta)}=\cos\vartheta_{kn}^{(\alpha,\beta)}$, $0<\vartheta_{kn}^{(\alpha,\beta)}<\pi$ of $P_n^{(\alpha,\beta)}(x)$ we have

$$-1<x_{nn}^{(\alpha,\beta)}<x_{n-1,n}^{(\alpha,\beta)}<\dots<x_{1n}^{(\alpha,\beta)}<1.$$

Let

$$\ell_{kn}^{(\alpha,\beta)}(x) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)\prime}(x_{kn})(x - x_{kn})} \qquad (k = 1, 2, \ldots, n).$$

For a fixed positive integer l, we define $Q_{nl}^{(\alpha,\beta)}(f, x)$ according to (4.1) and (4.2), where ℓ_k and f_k stand for $\ell_{kn}^{(\alpha,\beta)}(x)$ and $f(x_{kn}^{(\alpha,\beta)})$, respectively. As we noticed we have the properties analogous to (4.3) and (4.4) for $Q_{nl}^{(\alpha,\beta)}(f, x)$, too.

We proved in [12] (cf. [14])

Theorem 4.2. *Let l be a fixed positive integer, $n = 2lq$ $(q = 1, 2, \ldots)$ and $f \in C$. Then*

$$\lim_{n\to\infty} \|f(x) - Q_{nl}^{(\alpha,\beta)}(f, x)\| = 0$$

for any processes $Q_{nl}^{(\alpha,\beta)}$ supposing

$$-1 < \alpha, \beta < 0.5.$$

Our statement follows from the next more informative pointwise estimations

Theorem 4.3. *Let l be fixed natural number. Then for arbitrary fixed $\alpha, \beta > -1$ and $f \in C$*

$$\left|Q_{nl}^{(\alpha,\beta)}(f, x) - f(x)\right| = O(1) \sum_{i=1}^{n} \omega\left(f; \frac{\sqrt{1-x^2}}{n} i + \frac{i^2}{n^2}\right) \frac{1}{i^\gamma}$$

uniformly in n and $x \in [-1, 1]$, where $\gamma = \min(2; 1.5 - \alpha; 1.5 - \beta)$. $\big(\omega(f; t)$ is the modulus of continuity of $f(x)$.$\big)$

Remark 2. As it is well known, $Q_{nl}^{(-1/2,-1/2)}(f, x) = Q_{nl}(f, T, x)$. For this case some improvements of Theorem 4.2, including saturation, were proved recently by J. Szabados [11].

4.4

We define *the barycentric Bernstein-type operators* by

$$\mathcal{B}_n(f, \mathbf{e}_n, x) = \sum_{k=1}^{n/2} f\left(e_{2k-1,n}\right)\left\{b_{2k-1,n}(\mathbf{e}_n, x) + b_{2k,n}(\mathbf{e}_n, x)\right\}$$

(again, let n be even). In our forthcoming paper [13] among others we prove as follows

Theorem 4.4. *We have*

$$\lim_{n\to\infty} \|\mathcal{B}_n(f, \mathbf{e}_n, x) - f(x)\| = 0 \qquad \text{for any} \qquad f \in C.$$

Moreover the sequence $\{\mathcal{B}_n\}$ is saturated with the order $\{1/n\}$; the trivial class is the set of constant functions.

Remark 3. As far as we know the $\{\mathcal{B}_n\}$ process is *the first barycentric interpolation operator sequence which is convergent for arbitrary $f \in C$*; actually Theorem 4.4 can be proved for the sequences analogue to $\{Q_{nl}\}$ even if $l > 1$.

5 Some open problems and remarks

1. Statements analogous to Theorems 3.2, 3.3, 3.5 and 4.4 can be proved for the so called Floater–Hormann interpolants (see [4, §4]).

2. We intend to prove the Bernstein–Erdős conjecture in detail for the barycentric case. The interested reader may consult with [17, §3.3] for further orientation.

3. One can try to prove the almost everywhere divergence theorem of P. Erdős and P. Vértesi for the barycentric case (cf. [16, §2.5]).

4. Theorems for other "well-spaced" node systems (see [4, §6]) will be proved including the $\{x_{kn}^{(\alpha,\beta)}\}$ systems.

5. Probably it is quite difficult to get the saturation and saturation class for the operator sequences defined on other "well-spaced" nodes.

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