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Error Estimates for Polyharmonic Cubature Formulas

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Abstract

In the present article we shall present basic features of a polyharmonic cubature formula of degree s and corresponding error estimates. Main results are Markov-type error estimates for differentiable functions and error estimates for functions f which possess an analytic extension to a sufficiently large ball in the complex space $\mathbb{C}^d$.

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1 Introduction

Let $C\left(\mathbb{R}^d\right)$ be the set of all continuous complex-valued functions on the euclidean space $\mathbb{R}^d$. A *cubature formula* C is a linear functional on $C\left(\mathbb{R}^d\right)$ of the form

$$C(f) := \alpha_1 f(x_1) + ... + \alpha_N f(x_N). \tag{1}$$

The points $x_1, ..., x_N$ are called *nodes* or *knots* and the coefficients $\alpha_1, ..., \alpha_N \in \mathbb{R}$ the *weights*. A basic problem in numerical analysis is to approximate integrals of the form

$$\int f(x)\, d\mu(x)$$

for a (signed) measure μ in the euclidean space $\mathbb{R}^d$ by suitable cubature formulas.

An important characteristic of a cubature formula is exactness: the functional C is *exact on a subspace U of* $C\left(\mathbb{R}^d\right)$ with respect to a measure μ if

$$C(f) = \int f(x)\, d\mu(x) \tag{2}$$

holds for all $f \in U$. If U_s is the set of all polynomials $\mathcal{P}_s$ of degree $\leq s$, and the cubature is exact on U_s but not on U_{s+1}, we say that C has *order* s. Exactness on the space $\mathcal{P}_s$ can be expressed by the identities

$$C(x^\alpha) = \int x^\alpha\, d\mu(x)$$

for each multi-index $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d$ with $|\alpha| := \alpha_1 + ... + \alpha_d \leq s$ where $x^\alpha = x_1^{\alpha_1} ... x_d^{\alpha_d}$. In the theory of cubature formula it is assumed that the *moments*

$$\int x^\alpha\, d\mu(x)$$

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for $|\alpha| \leq s$ exist and that they can be explicitly calculated. The problem is to find constructive methods for determining nodes and weights from this information. In particular, a cubature formula leads to a solution of the so-called *truncated moment problem*. For a discussion of cubature formulas we refer to [26], [27], [29] and the recent survey [7].

In [15] and [18] we have introduced a new type of functional which approximates the integral

$$\int f(x)\,d\mu(x) \tag{3}$$

for a class of measures μ with support in the *ball*

$$B_R = \left\{x \in \mathbb{R}^d : |x| < R\right\} \tag{4}$$

and continuous functions $f : B_R \to \mathbb{C}$ where R is a positive number or ∞, and

$$r = |x| = \sqrt{x_1^2 + ... + x_d^2}$$

is the euclidean norm of $x = \left(x_1, ..., x_d\right) \in \mathbb{R}^d$. The unit sphere will be denoted by

$$\mathbb{S}^{d-1} := \left\{x \in \mathbb{R}^d : |x| = 1\right\},$$

and endowed with the rotation invariant measure $d\theta$.

Our approach is based on the Fourier-Laplace series of the function $f(x)$. In order to make concepts simpler we shall restrict our discussion in the introduction to the two-dimensional case where the Fourier-Laplace series is just the Fourier series of a function. Hence we define the basis functions

$$Y_{0,0}(x) = Y_{0,0}(r\cos t, r\sin t) = \frac{1}{\sqrt{2\pi}} \tag{5}$$

and

$$Y_{k,1}(x) = Y_{k,1}(r\cos t, r\sin t) = \frac{1}{\sqrt{\pi}} r^k \cos kt \tag{6}$$

$$Y_{k,2}(x) = Y_{k,2}(r\cos t, r\sin t) = \frac{1}{\sqrt{\pi}} r^k \sin kt \tag{7}$$

for $k \in \mathbb{N}$ where $\mathbb{N}$ denotes the set of all natural numbers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. A point $x \in \mathbb{R}^2$ is written as $x = (r\cos t, r\sin t)$ where r is the radius of x and $(\cos t, \sin t)$ is in the unit sphere. The Fourier coefficients of a continuous function f are defined by

$$f_{k,\ell}(r) = \int_0^{2\pi} f(r\cos t, r\sin t) \cdot Y_{k,\ell}(\cos t, \sin t)\,dt.$$

The *Fourier series* of the continuous function $f : B_R \to \mathbb{C}$ is defined by the formal expansion

$$\sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} f_{k,\ell}(r)\,Y_{k,\ell}(\theta) \tag{8}$$

where $a_0 = 1$ and $a_k = 2$ for $k \in \mathbb{N}$, and $\theta = (\cos t, \sin t)$. It is easy to see that $f_{k,\ell}$ is a continuous function if f is continuous. Furthermore, if f is infinitely differentiable in B_R then the function

$$f_{k,\ell}(r)\,r^{-k}$$

is *even* (and infinitely differentiable), see [6]. Finally, if f is a polynomial then $f_{k,\ell}(r)\,r^{-k}$ is a univariate polynomial in r^2, see Section 2 for more details.

If f is sufficiently smooth then the Fourier series (8) converges absolutely and uniformly on compact subsets of B_R to the function $f(x)$ and one obtains that

$$\begin{aligned}\int_{\mathbb{R}^2} f(x)\,d\mu &= \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \int_{\mathbb{R}^2} f_{k,\ell}(r)\,Y_{k,\ell}(\theta)\,d\mu(x) \\ &= \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \int_{\mathbb{R}^2} f_{k,\ell}(r)\,r^{-k} Y_{k,\ell}(x)\,d\mu(x).\end{aligned}$$

We shall now call a signed measure μ with support in $B_R \subset \mathbb{R}^2$ *pseudo-positive* if the inequality

$$\int_{\mathbb{R}^2} h(|x|) Y_{k,\ell}(x) \, d\mu(x) \geq 0$$

holds for every non-negative continuous function $h : [0,R] \to [0,\infty)$ and for all $k \in \mathbb{N}_0$, and $\ell = 1,...,a_k$. By the Riesz representation theorem there exist unique non-negative measures $\mu_{k,\ell}$ defined on $[0,R]$, which we call *component measures*, such that

$$\int_0^\infty h(t) \, d\mu_{k,\ell}(t) = \int_{\mathbb{R}^2} h(|x|) Y_{k,\ell}(x) \, d\mu$$

holds for all $h \in C[0,R]$. Using this notation we obtain

$$\int_{\mathbb{R}^2} f(x) \, d\mu = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \int_0^\infty f_{k,\ell}(r) r^{-k} d\mu_{k,\ell}(r).$$

In passing, we mention that radially symmetric measures are pseudo-positive.

The main idea in our approach is to use quadrature formulas to approximate the univariate integrals

$$\int_0^\infty f_{k,\ell}(r) r^{-k} d\mu_{k,\ell}(r). \tag{9}$$

Thus we assume in our approach that the Fourier coefficients $f_{k,\ell}(r)$ are known. One may use Fast-Fourier Transform to find approximations of $f_{k,\ell}$ and to combine these with our approach in order to find cubature formulas only involving the function values of f – a topic which we want to consider in a future paper.

Next we want to discuss which kind of quadrature formulas for approximating (9) are useful. Due to the fact that $f_{k,\ell}(r) r^{-k}$ is an even function for smooth f we shall require that the quadrature formula is exact for all polynomials of the form r^{2j} for $j = 0,...,2s-1$ where s a given natural number. By taking the transformation $\sqrt{r}$ this means that the transformed quadrature formula should be exact for all polynomial t^j for $j = 0,...,2s-1$ – and here the classical Gauß-Jacobi quadrature enters the game.

Our polyharmonic cubature formula is now defined in the following way: given a pseudo-positive measure μ we consider the component measures $\mu_{k,\ell}(r)$. Let $\mu_{k,\ell}^{\psi}$ be the image measure of $\mu_{k,\ell}$ for the transformation $\psi : [0,\infty) \to [0,\infty)$ defined by $\psi(r) = r^2$, so

$$\int_0^\infty f_{k,\ell}(r) r^{-k} d\mu_{k,\ell}(r) = \int_0^\infty f_{k,\ell}\left(\sqrt{t}\right) t^{-k/2} d\mu_{k,\ell}^{\psi}(t).$$

For the non-negative univariate measures $\mu_{k,\ell}^{\psi}$ we shall use the univariate Gauß-Jacobi quadratures $\nu_{k,\ell}^{(s)}$ of order $2s-1$ as an approximation of $\mu_{k,\ell}^{\psi}$. The *polyharmonic cubature* $T^{(s)}(f)$ *of degree* s is then defined by

$$T^{(s)}(f) := \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \int_0^\infty f_{k,\ell}\left(\sqrt{t}\right) t^{-\frac{1}{2}k} d\nu_{k,\ell}^{(s)}(t).$$

The cubature formula $T^{(s)}$ will be defined at first only for polynomials: then the sum in the definition of $T^{(s)}(f)$ is actually a finite sum and no convergence questions occur. The cubature formula $T^{(s)}$ has the property that

$$T^{(s)}\left(|x|^{2j} Y_{k,\ell}(x)\right) = \int |x|^{2j} Y_{k,\ell}(x) \, d\mu(x)$$

for all $j = 0,...,2s-1$ and for all $k \in \mathbb{N}_0$, $\ell = 1,...,a_k$. This is equivalent to the functional $T^{(s)}$ being exact on the space of all polynomials of polyharmonic order $\leq 2s$.

In [15] we investigated the truncated moment problem for pseudo-positive measures. In the present article we shall present a Markov-type error estimate for the polyharmonic cubature formula and apply this estimate to functions f which possess an analytic extension on the ball in $\mathbb{C}^d$ with center 0 and sufficiently large radius. For an error estimate of polyharmonic cubature formula based on complex methods we refer to [16]. As general background information we mention as well our unpublished manuscript [18] which contains also instructive examples.

The paper is organized in the following way: in Section 2 we shall provide background material about spherical harmonics and Fourier-Laplace series which is necessary for the case $d > 2$. In Section 3 we give a short review of properties of the polyharmonic cubature formulas. Section 4 contains the main result of the paper – an error estimate for $T^{(s)}$ which is based on the error estimate of Markov for quadratures.

2 Polyharmonic polynomials and Spherical harmonics

We shall write $x \in \mathbb{R}^d$ in spherical coordinates $x = r\theta$ with $\theta \in \mathbb{S}^{d-1}$. Let $\mathcal{H}_k\left(\mathbb{R}^d\right)$ be the set of all harmonic homogeneous complex-valued polynomials of degree k. Then $f \in \mathcal{H}_k\left(\mathbb{R}^d\right)$ is called a *solid harmonic* and the restriction of f to $\mathbb{S}^{d-1}$ a *spherical harmonic* of degree k and we set

$$a_k := \dim \mathcal{H}_k\left(\mathbb{R}^d\right), \tag{10}$$

see [28], [25], [1], [13] for details. Throughout the paper we shall assume

$$Y_{k,\ell} : \mathbb{R}^d \to \mathbb{R}, \ell = 1, ..., a_k, \tag{11}$$

is an *orthonormal basis* of $\mathcal{H}_k\left(\mathbb{R}^d\right)$ with respect to the scalar product

$$\langle f, g\rangle_{\mathbb{S}^{d-1}} := \int_{\mathbb{S}^{d-1}} f(\theta)\overline{g(\theta)}d\theta.$$

We shall often use the trivial identity $Y_{k,\ell}(x) = r^k Y_{k\ell}(\theta)$ for $x = r\theta$. Further we define the surface area ω_d by

$$\omega_d = \int_{\mathbb{S}^{d-1}} 1 d\theta.$$

The *Fourier-Laplace series* of the continuous function $f : B_R \to \mathbb{C}$, is defined by the formal expansion

$$f(r\theta) = \sum_{k=0}^{\infty}\sum_{\ell=1}^{a_k} f_{k,\ell}(r) Y_{k,\ell}(\theta) \tag{12}$$

where a_k is defined in (10) and the *Fourier-Laplace coefficient* $f_{k,\ell}(r)$ is defined by

$$f_{k,\ell}(r) = \int_{\mathbb{S}^{d-1}} f(r\theta) Y_{k,\ell}(\theta) d\theta \tag{13}$$

for any non-negative real number r with $0 \le r < R$.

There is a strong interplay between algebraic and analytic properties of the function f and those of the Fourier-Laplace coefficients $f_{k,\ell}$. For example, if $f(x)$ is a polynomial in the variable $x = (x_1, ..., x_d)$ then the Fourier-Laplace coefficient $f_{k,\ell}$ is of the form $f_{k,\ell}(r) = r^k p_{k,\ell}(r^2)$ where $p_{k,\ell}$ is a univariate polynomial, see e.g. in [28] or [26]. Hence, the *Fourier-Laplace* series (12) of a polynomial $f(x)$ is equal to

$$f(x) = \sum_{k=0}^{\deg f}\sum_{\ell=1}^{a_k} p_{k,\ell}(|x|^2) Y_{k,\ell}(x) \tag{14}$$

where $\deg f$ is the total degree of f and $p_{k,\ell}$ is a univariate polynomial of degree $\le \deg f - k$. This representation is often called the *Gauss representation*.

A similar formula is valid for a much larger class of functions. Let us recall that a function $f : G \to \mathbb{C}$ defined on an open set G in $\mathbb{R}^d$ is called *polyharmonic of order* N if f is $2N$ times continuously differentiable and

$$\Delta^N u(x) = 0 \tag{15}$$

for all $x \in G$ where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_d^2}$$

is the Laplace operator and Δ^N the N-th iterate of Δ. The theorem of Almansi states that for a polyharmonic function f of order N defined on the ball $B_R = \left\{x \in \mathbb{R}^d : |x| < R\right\}$ there exist univariate polynomials $p_{k,\ell}(t)$ of degree $\le N - 1$ such that

$$f(x) = \sum_{k=0}^{\infty}\sum_{\ell=1}^{a_k} p_{k,\ell}(|x|^2) Y_{k,\ell}(x) \tag{16}$$

where convergence of the sum is uniform on compact subsets of B_R, see e.g. [26], [3], [2] and [17] for further extensions. Neglecting at the moment questions of convergence we see that

$$\int f(x)\,d\mu(x) = \sum_{k=0}^{\infty}\sum_{\ell=1}^{a_k}\int p_{k,\ell}(|x|^2)Y_{k,\ell}(x)\,d\mu(x).$$

Note that $p_{k,\ell}$ is a univariate function depending on $|x|^2$ and note that $|x|^{2s}\,Y_{k,\ell}(x)$ is indeed a polynomial and therefore

$$\int |x|^{2s}\,Y_{k,\ell}(x)\,d\mu(x)$$

can be expressed as a sum of monomial moments. The above mentioned Gauss decomposition just says that each multivariate polynomial $f(x)$ is indeed a linear combination of polynomials of the type $|x|^{2s}\,Y_{k,\ell}(x)$.

These considerations have led us to the following definition: a signed measure μ with support in $B_R \subset \mathbb{R}^d$ is *pseudo-positive with respect to the orthonormal basis* $Y_{k,\ell}, \ell = 1,\dots,a_k$, $k \in \mathbb{N}_0$ if the inequality

$$\int_{\mathbb{R}^d} h(|x|)\,Y_{k,\ell}(x)\,d\mu(x) \geq 0 \tag{17}$$

holds for every non-negative continuous function $h : [0,R] \to [0,\infty)$ and for all $k \in \mathbb{N}_0$, $\ell = 1,2,\dots,a_k$. Then the following can be proved, see [15].

Theorem 2.1. *Let μ be a pseudo-positive measure on $\mathbb{R}^d$ with support in $B_R \subset \mathbb{R}^d$. Then there exist unique non-negative measures $\mu_{k,\ell}$ with support in $[0,R]$, which we call* component measures*, such that*

$$\int_0^{\infty} h(t)\,d\mu_{k,\ell}(t) = \int_{\mathbb{R}^d} h(|x|)\,Y_{k,\ell}(x)\,d\mu \tag{18}$$

holds for all $h \in C[0,R]$. Further

$$\int_{\mathbb{R}^d} f(x)\,d\mu = \sum_{k=0}^{\infty}\sum_{\ell=1}^{a_k}\int_0^{\infty} f_{k,\ell}(r)\,r^{-k}\,d\mu_{k,\ell}(r) \tag{19}$$

for each $f \in C\left(\mathbb{R}^d\right)$ whose Fourier-Laplace series has only finitely many non-zero terms.

Let $\psi : [0,\infty) \to [0,\infty)$ be the transformation $\psi(t) = t^2$ and let $\mu_{k,\ell}^{\psi}$ be the image measure of $\mu_{k,\ell}$ under ψ. Then (19) becomes

$$\int_{\mathbb{R}^d} f(x)\,d\mu = \sum_{k=0}^{\infty}\sum_{\ell=1}^{a_k}\int_0^{\infty} f_{k,\ell}\left(\sqrt{t}\right)t^{-\frac{1}{2}k}\,d\mu_{k,\ell}^{\psi}(t). \tag{20}$$

The *main idea* is simple and consists in replacing in formula (20) the non-negative univariate measures $\mu_{k,\ell}^{\psi}$ by their univariate Gauß-Jacobi quadratures $\nu_{k,\ell}^{(s)}$ of order $2s-1$. Then we obtain a functional $T^{(s)}$ defined on the set $\mathbb{C}\left[x_1,x_2,\dots,x_d\right]$ of all polynomials by setting

$$T^{(s)}(f) := \sum_{k=0}^{\infty}\sum_{\ell=1}^{a_k}\int_0^{\infty} f_{k,\ell}\left(\sqrt{t}\right)t^{-\frac{1}{2}k}\,d\nu_{k,\ell}^{(s)}(t). \tag{21}$$

Since f is a polynomial the series is finite and therefore $T^{(s)}$ is well-defined.

Sometimes it is useful to rewrite the definition of $T^{(s)}(f)$ using the variable r instead of t. If we define $\psi^{-1}(t) = \sqrt{t}$ (so ψ^{-1} is the inverse function of ψ) and if $\sigma_{k,\ell}^{(s)}$ is the image measure of $\nu_{k,\ell}^{(s)}$ under ψ^{-1}, then we may write

$$T^{(s)}(f) = \sum_{k=0}^{\infty}\sum_{\ell=1}^{a_k}\int_0^{\infty} f_{k,\ell}(r)\,r^{-k}\,d\sigma_{k,\ell}^{(s)}(r).$$

3 Basic properties of the polyharmonic cubature

We shall recall from [15] and [18] some basic properties for the polyharmonic cubature formula:

Theorem 3.1. *Let μ be a pseudo-positive measure with support in the ball B_R. Then the functional $T^{(s)} : \mathbb{C}[x_1, x_2, ..., x_d] \to \mathbb{C}$ is continuous with respect to the supremum norm provided that the* summability assumption

$$\sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \int_0^{\infty} r^{-k} d\mu_{k,\ell}(r) < \infty \tag{22}$$

holds.

Proof. Since μ has support in B_R the measures $\mu_{k,\ell}$ have support in $[0, R]$. For the Fourier-Laplace coefficient $f_{k,\ell}$ we have

$$\left| f_{k,\ell}(r) \right| \leq C \max_{|x| \leq R} \left| f(x) \right| \quad \text{for } 0 \leq r \leq R.$$

Hence

$$\left| \int_0^{\infty} f_{k,\ell}(r) r^{-k} d\sigma_{k,\ell}^{(s)}(r) \right| \leq C \max_{|x| \leq R} \left| f(x) \right| \int_0^{\infty} r^{-k} d\sigma_{k,\ell}^{(s)}(r)$$

and

$$\left| T^{(s)}(f) \right| \leq C \max_{|x| \leq R} \left| f(x) \right| \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \int_0^{\infty} r^{-k} d\sigma_{k,\ell}^{(s)}(r). \tag{23}$$

For the convergence in (23) it suffices to prove

$$\int_0^{\infty} r^{-k} d\sigma_{k,\ell}^{(s)}(r) \leq \int_0^{\infty} r^{-k} d\mu_{k,\ell}(r). \tag{24}$$

This inequality follows from the extremal property of the Gauß–Jacobi quadrature, see Theorem 4.1 in Chapter 4 of [21]. □

By the Riesz representation theorem there exists a signed measure $\sigma^{(s)}$ with support in the closed ball B_R such that

$$T^{(s)}(f) = \int_{B_R} f(x) d\sigma^{(s)}(x)$$

for all continuous functions $f : B_R \to \mathbb{C}$. Moreover, the component measures of the pseudo–positive measure $\sigma^{(s)}$ are exactly the univariate measures $\sigma_{k,\ell}^{(s)}$.

Note that the summability condition (22) can be rephrased in terms of the measure μ by the identity

$$\int_0^{\infty} r^{-k} d\mu_{k,\ell}(r) = \int_{\mathbb{R}^d} Y_{k,\ell}\left(\frac{x}{|x|}\right) d\mu.$$

We summarize the results in the following

Theorem 3.2. *Let μ be a pseudo-positive signed measure with support in the closed ball B_R satisfying the summability condition (22). Then for each natural number s there exists a unique pseudo-positive, signed measure $\sigma^{(s)}$ with support in B_R such that*

(i) The support of each component measure $\sigma_{k,\ell}^{(s)}$ of $\sigma^{(s)}$ has cardinality $\leq s$.

(ii) $\int P d\mu = \int P d\sigma^{(s)}$ for all polynomials P with $\Delta^{2s} P = 0$.

Proof. The exactness of the Gauß-Jacobi quadratures $\nu_{k,\ell}^{(s)}$ for polynomials of degree $\leq 2s - 1$ implies that $T^{(s)}$ and μ coincide on the set of all polynomials P such that $\Delta^{2s} P = 0$. This is due to the fact that in the Laplace–Fourier expansion the coefficients are given by $f_{k,\ell}(r) = r^k p_{k,\ell}(r^2)$ where $p_{k,\ell}$ are polynomials of degree $2s - 1$. □

Definition 3.1. The measure $\sigma^{(s)}$ constructed in the last Theorem will be called the **polyharmonic Gauß-Jacobi measure of order** s for the measure μ.

The following is an analog to the theorem of Stieltjes about the convergence of the univariate Gauß–Jacobi quadrature formulas.

Theorem 3.3. *Let $\sigma^{(s)}$ be the polyharmonic Gauß-Jacobi measure of order s for the measure μ, obtained in Theorem 3.2. Then*

$$\int f(x)\, d\sigma^{(s)} \rightarrow \int f(x)\, d\mu \qquad \text{for } s \rightarrow \infty$$

holds for every function $f \in C(B_R)$.

Proof. For any polynomial P the convergence $T^{(s)}(P) \rightarrow P$ holds for $s \longrightarrow \infty$. By standard results, the convergence $T^{(s)}(f) \rightarrow f$ carries over to all continuous functions $f : B_R \rightarrow \mathbb{C}$ provided there exists a constant $C > 0$ such that

$$\left| T^{(s)}(f) \right| \leq C \max_{|x| \leq R} \left| f(x) \right|.$$

for all natural numbers s and all $f \in C(B_R)$. □

In a similar way one can prove the following result:

Theorem 3.4. *Let μ be a pseudo-positive signed measure with support in B_R satisfying the summability condition (22) and let $\sigma^{(s)}$ be the polyharmonic Gauß-Jacobi measure of order s. If $f \in C^{2s}(\mathbb{R}^d)$ has the property that*

$$\frac{d^{2s}}{dt^{2s}} \left[f_{k,\ell}\left(\sqrt{t}\right) t^{-\frac{1}{2}k} \right] \geq 0,$$

for all $t \in (0, R^2)$ and for all $k \in \mathbb{N}_0$, $\ell = 1, 2, ..., a_k$, then the following inequality

$$\int f(x)\, d\sigma^{(s)} \leq \int f(x)\, d\mu$$

holds.

Let us note that every signed measure $d\mu$ with bounded variation may be represented (non-uniquely) as a difference of two pseudo-positive measures. We refer to [15] for instructive examples of pseudo-positive measures.

4 Error estimate of the Polyharmonic Gauss-Jacobi Cubature formula

The topic of estimation of quadrature formulas for smooth and analytic functions is a widely studied one. Beyond the classical monographs [22], [8, p. 344], [9], we provide further and more recent publications, as [5], [10], [11], [12], [20], [23].

We recall here the following error estimate of Markov:

Theorem 4.1. *(Markov) Let ν be a non-negative measure over the interval $[a, b]$ and let ν^s be the Gauss-Jacobi measure of order s. Define for every $g \in C[a, b]$ the error*

$$E_s(g) := \int_a^b g(t)\, d\nu(t) - \int_a^b g(t)\, d\nu^{(s)}(t).$$

If $g \in C^{2s}[a, b]$ then

$$\left| E_s(g) \right| \leq \frac{1}{(2s)!} \sup_{a<\xi<b} \left| g^{(2s)}(\xi) \right| \int_a^b \left| Q^s(t) \right|^2 d\nu(t)$$

where $Q^s(t)$ is the orthogonal polynomial of degree s, with leading coefficient 1, relative to ν.

We shall prove now the following analogue:

Theorem 4.2. *Let $0 < R < \infty$ and let $\psi : [0,\infty) \to [0,\infty)$ be defined by $\psi(t) = t^2$. Let μ be a pseudo-positive signed measure with support in B_R satisfying the summability condition (22), and let $\sigma^{(s)}$ be the polyharmonic Gauß-Jacobi measure of order s. Define for every $f \in C(B_R)$ the error functional*

$$E_s(f) := \int f(x)\,d\mu(x) - \int f(x)\,d\sigma^{(s)}(x).$$

If $f \in C^{2s}(B_R) \cap C\left(\overline{B_R}\right)$ then the error $E_s(f)$ is less than or equal to

$$\frac{1}{(2s)!}\sum_{k=0}^{\infty}\sum_{\ell=1}^{a_k}\sup_{0<\xi<R^2}\left|\frac{d^{2s}}{dt^{2s}}\left[f_{k,\ell}\left(\sqrt{t}\right)t^{-\frac{1}{2}k}\right](\xi)\right|\int_0^{R^2}\left|Q^s_{k,\ell}(t)\right|^2 d\mu^{\psi}_{k\ell}.$$

Here $Q^s_{k,\ell}(t)$ is the orthogonal polynomial of degree s with respect to the measure $\mu^{\psi}_{k\ell}$, having a leading coefficient equal to 1; if the support of $\mu_{k,\ell}$ has less than s points, $Q^s_{k,\ell}$ is defined to be 0.

Proof. Since $f \in C^{2s}(B_R) \cap C\left(\overline{B_R}\right)$ it is easy to see that the Fourier-Laplace coefficients $f_{k,\ell} \in C^{2s}(0,R) \cap C[0,R]$. Let $\mu_{k,\ell}$ and $\sigma_{k,\ell}$, $k \in \mathbb{N}_0$, $\ell = 1,\ldots,a_k$, and $\sigma^{(s)}$ be as in Theorem 3.2. From the definitions it follows

$$E_s(f) = \sum_{k=0}^{\infty}\sum_{\ell=1}^{a_k}\int_0^R f_{k,\ell}(r)\,r^{-k}d\mu_{k,\ell} - \int_0^R f_{k,\ell}(r)\,r^{-k}d\sigma^{(s)}_{k,\ell}.$$

Further $f_{k,\ell}(r)\,r^{-k}$ is integrable with respect to $\mu_{k,\ell}$ since $f_{k,\ell}$ is continuous on $[0,R]$ and condition (22) holds. Let us fix the pair of indices (k,ℓ). If the support of $\mu_{k,\ell}$ has less than s points we know that $\mu_{k,\ell} = \sigma^{(s)}_{k,\ell}$. So assume that the support of $\mu_{k,\ell}$ has at least s points. Then the support of $\mu^{\psi}_{k,\ell}$ has at least s points and in our construction $\nu^{(s)}_{k,\ell}$ is the Gauß-Jacobi measure of $\mu^{\psi}_{k,\ell}$. Consequently

$$\begin{aligned} e\left(f_{k,\ell}\right) &:= \int_0^R f_{k,\ell}(r)\,r^{-k}d\mu_{k,\ell}(r) - \int_0^R f_{k,\ell}(r)\,r^{-k}d\sigma^{(s)}_{k,\ell}(r) \\ &= \int_0^{R^2} f_{k,\ell}\left(\sqrt{t}\right)t^{-\frac{1}{2}k}d\mu^{\psi}_{k,\ell}(t) - \int_0^{R^2} f_{k,\ell}\left(\sqrt{t}\right)t^{-\frac{1}{2}k}d\nu^{(s)}_{k,\ell}(t). \end{aligned}$$

By Markov's error estimate one obtains with $g_{k,\ell}(t) := f_{k,\ell}\left(\sqrt{t}\right)t^{-\frac{1}{2}k}$ the inequality

$$e\left(f_{k,\ell}\right) \le \frac{1}{(2s)!}\sup_{0<\xi<R^2}\left|g^{(2s)}_{k,\ell}(\xi)\right|\int_{0^2}^{R^2}\left|Q^s_{k,\ell}(t)\right|^2 d\mu^{\psi}_{k,\ell}(t).$$

The proof is complete. □

Now we are going to apply the results for holomorphic functions in several variables. We define the complex ball in $\mathbb{C}^d$ with center 0 and radius τ by

$$B^{\mathbb{C}}_{\tau} = \{(w_1,\ldots,w_d) \in \mathbb{C}^d : \sum_{j=1}^{d}\left|w_j\right|^2 < \tau^2\}.$$

We assume that f is holomorphic on $B^{\mathbb{C}}_{\tau}$ for $\tau > R$. For fixed $\theta \in \mathbb{S}^{d-1}$ we define a map

$$\varphi_\theta : \{z \in \mathbb{C} : |z| < \tau\} \to B^{\mathbb{C}}_{\tau} \text{ by } \varphi_\theta(z) = z\theta$$

which is clearly holomorphic. Hence f_θ defined by $f_\theta(z) = f(z\theta) = f \circ \varphi_\theta(z)$ is holomorphic. It follows that $f_{k,\ell}(z)$ defined by

$$f_{k,\ell}(z) = \int_{\mathbb{S}^{d-1}} f(z\theta)\,Y_{k,\ell}(\theta)\,d\theta \tag{25}$$

is a holomorphic extension of $f_{k,\ell}$ to $\{z \in \mathbb{C} : |z| < \tau\}$. For further material about analytic extensions of Fourier-Laplace series and Fourier-Laplace coefficients we refer to [14], [17] and [24].

Now we need the following result:

Lemma 4.3. *Let f be a holomorphic function on the open ball $B_\tau^{\mathbb{C}}$ for $\tau > 0$. Let $f_{k,\ell}$ be the Fourier-Laplace coefficient of f and define*

$$p_{k,\ell}(t) = f_{k,\ell}\left(\sqrt{t}\right) \cdot t^{-k/2}$$

for $0 < t < \tau^2$ Then the following inequality

$$\left|\frac{d^s}{dt^s} p_{k,\ell}(t)\right| \leq \sqrt{\omega_d} \max_{u\in[0,2\pi],\theta\in\mathbb{S}^{d-1}} \left|f\left(e^{iu}\rho\theta\right)\right| \frac{\rho^{2-k}s!}{\left(\rho^2 - t\right)^{s+1}} \tag{26}$$

holds for all $0 < t < \rho^2 < \tau^2$ and for all natural numbers s.

Proof. We apply Cauchy-Schwarz inequality to the integral (25) obtaining

$$\left|f_{k,\ell}(z)\right|^2 \leq \int_{\mathbb{S}^{d-1}} \left|f(z\theta)\right|^2 d\theta \cdot \int_{\mathbb{S}^{d-1}} \left|Y_{k,\ell}(\theta)\right|^2 d\theta.$$

Since $Y_{k,\ell}$ is orthonormal we obtain for $z = |z| e^{iu}$ and $|z| = \rho$

$$\left|f_{k,\ell}(z)\right|^2 \leq \omega_d \max_{u\in[0,2\pi],\theta\in\mathbb{S}^{d-1}} \left|f\left(e^{iu}\rho\theta\right)\right|^2. \tag{27}$$

Let us recall the Cauchy estimates for a holomorphic function g in the ball $|z| < \tau$ applied for $|z| = \rho$

$$\left|g^{(n)}(0)\right| \leq \frac{n!}{\rho^n} \max_{|z|=\rho} \left|g(z)\right|$$

We apply this estimate to the holomorphic function $f_{k,\ell}(z)$ and $n = m + k$ and we use (27):

$$\left|\frac{d^{m+k}}{dz^{m+k}} f_{k,\ell}(0)\right| \leq \frac{(k+m)!}{\rho^{m+k}} \sqrt{\omega_d} \max_{u\in[0,2\pi],\theta\in\mathbb{S}^{d-1}} \left|f\left(e^{iu}\rho\theta\right)\right|. \tag{28}$$

Since $f_{k,\ell}(z)$ is holomorphic for $|z| < \tau$ we can write $f_{k,\ell}$ as a power series. Further it is known (see [6]) that

$$f_{k,\ell}^{(j)}(0) = 0 \text{ for } j = 0, ..., k-1,$$

Hence we can write for $|z| < \tau$

$$f_{k,\ell}(z) = \sum_{m=k}^{\infty} \frac{1}{m!} \frac{d^m}{dr^m} f_{k,\ell}(0) \cdot z^m.$$

It is known that $r^{-k} f_{k,\ell}(r)$ is an even function (see [6]), hence we can obtain a description for the function $p_{k,\ell}\left(r^2\right)$:

$$p_{k,\ell}\left(r^2\right) = r^{-k} f_{k,\ell}(r) = \sum_{m=0}^{\infty} \frac{1}{(k+2m)!} \frac{d^{2m+k}}{dr^{2m+k}} f_{k,\ell}(0) \cdot r^{2m}.$$

Then for $t = r^2$ we conclude that

$$p_{k,\ell}(t) = \sum_{m=0}^{\infty} \frac{1}{(k+2m)!} \frac{d^{2m+k}}{dr^{2m+k}} f_{k,\ell}(0) \cdot t^m.$$

We infer that

$$\frac{d^s}{dt^s} p_{k,\ell}(t) = \sum_{m=s}^{\infty} \frac{1}{(k+2m)!} \frac{m!}{(m-s)!} \frac{d^{2m+k}}{dr^{2m+k}} f_{k,\ell}(0) \cdot t^{(m-s)}.$$

Now (28) implies

$$\left|\frac{d^s}{dt^s} p_{k,\ell}(t)\right| \leq \sqrt{\omega_d} \max_{u\in[0,2\pi],\theta\in\mathbb{S}^{d-1}} \left|f\left(e^{iu}\rho\theta\right)\right| \frac{1}{\rho^{k+2s}} \sum_{m=s}^{\infty} \frac{m!}{(m-s)!} \left(\frac{t}{\rho^2}\right)^{m-s}.$$

For $|x| < 1$ we have

$$\sum_{m=s}^{\infty} \frac{m!}{(m-s)!} x^{m-s} = \frac{d^s}{dx^s} \sum_{m=0}^{\infty} x^m = \frac{d^s}{dt^s} \frac{1}{1-x} = s!\,(1-x)^{-s-1}$$

and we see that

$$\left| \frac{d^s}{dt^s} p_{k,\ell}(t) \right| \leq \sqrt{\omega_d} \max_{u\in[0,2\pi],\theta\in\mathbb{S}^{d-1}} \left| f\left(e^{iu}\rho\theta\right) \right| \frac{s!}{\rho^{k+2s}} \left(1 - \frac{t}{\rho^2}\right)^{-s-1}$$

which gives (26). □

Combining the last two results we obtain:

Theorem 4.4. *Let μ be a pseudo-positive signed measure with support in B_R satisfying the summability condition (22) and let $\sigma^{(s)}$ be the polyharmonic Gauß-Jacobi measure of order s. Then the error $E_s(f)$ is less than or equal to*

$$\frac{\sqrt{\omega_d}\rho^2}{\left(\rho^2 - R^2\right)^{2s+1}} \max_{w\in\mathbb{C}^n, |w|\leq\rho} |f(w)| \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \frac{1}{\rho^k} \int_0^{R^2} \left| Q_{k,\ell}^s(t) \right|^2 d\mu_{k,\ell}^{\psi}(t)$$

for all functions $f : B_R \to \mathbb{C}$ which possess a holomorphic extension to the complex ball $B_\tau^{\mathbb{C}}$ for $\tau > R$ where ρ is any number with $R < \rho < \tau$.

Proof. By Theorem 4.2 the error $E_s(f)$ is less than or equal

$$\frac{1}{(2s)!} \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \sup_{0<\xi<R^2} \left| \frac{d^{2s}}{dt^{2s}} \left[f_{k,\ell}\left(\sqrt{t}\right) t^{-\frac{1}{2}k} \right](\xi) \right| \int_0^{R^2} \left| Q_{k,\ell}^s(t) \right|^2 d\mu_{k\ell}^{\psi}.$$

Lemma 4.3 applied for index $2s$ and for $p_{k,\ell}(t) = f_{k,\ell}\left(\sqrt{t}\right) t^{-\frac{1}{2}k}$ shows that

$$\left| \frac{d^{2s}}{dt^{2s}} p_{k,\ell}(t) \right| \leq \sqrt{\omega_d} \max_{u\in[0,2\pi],\theta\in\mathbb{S}^{d-1}} \left| f\left(e^{iu}\rho\theta\right) \right| \frac{\rho^{2-k}(2s)!}{\left(\rho^2 - t\right)^{2s+1}}.$$

Using that $\rho^2 - \xi \geq R^2$ for $0 < \xi < R^2$ we conclude that the error $E_s(f)$ is less than or equal

$$\sqrt{\omega_d} \max_{u\in[0,2\pi],\theta\in\mathbb{S}^{d-1}} \left| f\left(e^{iu}\rho\theta\right) \right| \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \frac{\rho^{2-k}}{\left(\rho^2 - R^2\right)^{2s+1}} \int_0^{R^2} \left| Q_{k,\ell}^s(t) \right|^2 d\mu_{k\ell}^{\psi}$$

and the statement is proven. □

We can simplify the estimate in the following way:

Theorem 4.5. *Let μ be a pseudo-positive signed measure with support in B_R satisfying the summability condition (22) and let $\sigma^{(s)}$ be the polyharmonic Gauß-Jacobi measure of order s. Then the error $E_s(f)$ is less than or equal to*

$$\frac{\sqrt{\omega_d}\rho^2 R^{2s}}{\left(\rho^2 - R^2\right)^{2s+1}} \max_{w\in\mathbb{C}^n, |w|\leq\rho} |f(w)| \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \left(\frac{R}{\rho}\right)^k \int_0^R r^{-k} d\mu_{k,\ell}(r)$$

for all functions $f : B_R \to \mathbb{C}$ which possess a holomorphic extension to the complex ball $B_\tau^{\mathbb{C}}$ for $\tau > R$ where ρ is any number with $R < \rho < \tau$.

Proof. Note that the polynomial $Q_{k,\ell}^s(t)$ of degree s is of the form

$$Q_{k,\ell}^s(t) = \left(t - t_{1,k,\ell}\right) .... \left(t - t_{s,k,\ell}\right)$$

where the points $t_{j,k,\ell}$ are in the interval $\left(0, R^2\right)$. It follows that $\left|t - t_{j,k.\ell}\right| < R^2$ and we obtain the estimate

$$\int_0^{R^2} \left| Q_{k,\ell}^s(t) \right|^2 d\mu_{k,\ell}^{\psi}(t) \leq R^{2s} \int_0^{R^2} 1 d\mu_{k,\ell}^{\psi}(t) = R^{2s} \int_0^R 1 d\mu_{k,\ell}(r).$$

Since

$$\int_0^R 1d\mu_{k,\ell}(r) = \int_0^R r^k r^{-k} d\mu_{k,\ell}(r) \leq R^k \int_0^R r^{-k} d\mu_{k,\ell}(r)$$

we can finally estimate

$$\frac{1}{\rho^k} \int_0^{R^2} \left| Q^s_{k,\ell}(t) \right|^2 d\mu^{\psi}_{k,\ell}(t) \leq R^{2s} \left(\frac{R}{\rho} \right)^k \int_0^R r^{-k} d\mu_{k,\ell}(r)$$

and in view of Theorem 4.4 the statement is proved. □

Finally we see that

$$\frac{R^2}{\rho^2 - R^2} < 1$$

is equivalent to the condition $2R^2 < \rho^2$. Thus for functions f which have a holomorphic extension to the complex ball with radius $\tau > 2R^2$ we obtain an estimate where the error decreases rapidly when the order of the polyharmonic cubature is increased.

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