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A numerical approach for a special crack problem

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Abstract

A collocation-quadrature method is proposed and studied for the numerical solution of a singular integral equation concerned with the two-dimensional elasticity problem of a crack at a circular cavity surface. These investigations are based on an C^*-algebra approach presented in a recent paper [5] on the numerical solution of an integral equation for the notched half-plane problem.

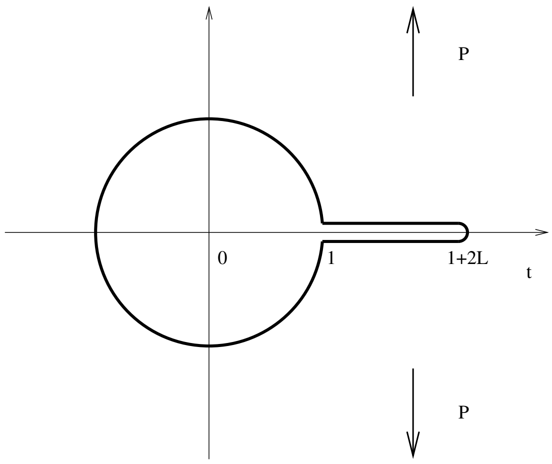
1 Introduction

In [11, (37.5)], the integral equation

$$\frac{1}{\pi}\int_1^{1+2L}\left[\frac{1}{s-t}+k_0(t,s)\right]v_0(s)\,ds = f_0(t)+C\,, \quad 1<t<1+2L\,, \tag{1}$$

is given for studying the crack problem, which considers a circular hole of radius 1 and radial cut of length $2L$ at the surface of this hole in an elastic plane, which is subjected at infinity to tensile forces P perpendicular to the cut. Here,

$$k_0(t,s)=\frac{(t-s)(t^2-1)}{ts(ts-1)^3}-\frac{1}{s^3(ts-1)}-\frac{1}{ts^2} \quad \text{and} \quad f_0(t)=P\left(\frac{1}{4t^3}+\frac{1}{4t}-\frac{t}{2}\right) \tag{2}$$



The unknown function $v_0(t)$ of equation (1) measures the normal displacement of the face of the cut and has to satisfy the condition

$$v_0(1+2L)=0\,. \tag{3}$$

Also the constant $C\in\mathbb{R}$ is unknown. In case of $L=0.5$, equation (1) takes the form (cf. also [2, (14.7)])

$$\frac{1}{\pi}\int_0^1\left[\frac{1}{y-x}+k_0(1+x,1+y)\right]v(y)\,dy = f_0(1+x)+C\,, \quad 0<x<1\,, \tag{4}$$

together with the condition

$$v(1)=0\,, \tag{5}$$

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where $v(x)=v_0(1+x)$ and

$$k_0(1+x,1+y)=\frac{(x-y)x(x+2)}{(x+1)(y+1)(y+x+xy)^3}-\frac{1}{(y+1)^3(y+x+xy)}-\frac{1}{(x+1)(y+1)^2},$$

$$f_0(1+x)=P\left[\frac{1}{4(1+x)^3}+\frac{1}{4(1+x)}-\frac{1+x}{2}\right].$$

For the general case $L>0$, we get

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{y-x}+k_0^L(x,y)\right]v(y)\,dy=f_0^L(x)+C\,,\quad -1<x<1\,, \tag{6}$$

together with (5), $v(x)=v_0(1+L(1+x))$, and

$$k_0^L(x,y)=L\,k_0(1+L(1+x),1+L(1+y)),\quad f_0^L(x)=f_0(1+L(1+x)).$$

It turns out that the unknown constant C in (1) leads to problems in handling this operator equation as well analytically as numerically (cf. also the discussion in Section 4). For that reason, in Section 2 we transform equation (1) into an integral equation the unknown function of which is the derivative of the normal displacement function. In Section 3 we propose a collocation-quadrature method for the numerical solution of this integral equation and study the stability of this method, basing on a C^*-algebra approach for an integral equation of the notched half-plane problem presented in [5]. Section 4 contains a discussion of the numerical results obtained with the method of the present paper in comparison with results available from the literature. In Section 5, we give the technical proof of Lemma 2.2.

Note, that we do not loose information on the solution of (1) by transforming this equation into an equivalent one for the derivative of the normal displacement function $v_0(t)$. Of course, $v_0(t)$ can be recovered from its derivative by integration. Moreover, the important stress intensity factor at $t=1+2L$ can also be computed directly from $v_0'(t)$ (cf. (34)).

2 The integral equation for the derivative of the displacement

By elementary calculations one can see that, for $t,s>1$ and for

$$\widetilde{k}_0(t,s)=\frac{t}{ts-1}-\left(t+\frac{1}{t}-\frac{2}{t^3}\right)\frac{1}{(ts-1)^2}+\left(t-\frac{2}{t}+\frac{1}{t^3}\right)\frac{1}{(ts-1)^3}-\left(1+\frac{1}{t^2}\right)\left(\frac{1}{s}+\frac{1}{t}\right), \tag{7}$$

we have

$$\frac{\partial k_0(t,s)}{\partial t}=\frac{\partial \widetilde{k}_0(t,s)}{\partial s}\quad\text{and}\quad \widetilde{k}_0(t,1)=\frac{1}{1-t}\,. \tag{8}$$

We assume that the homogeneous equation (6) (i.e., $f_0^L(x)+C\equiv 0$) has only the trivial solution in the space $\bigcap_{1<p<\infty}\mathbf{L}^p(-1,1)$. Then, the following lemma holds ([2, Section 14, 2^o and Theorem 14.1]).

Lemma 2.1. *Equation (6) has a unique solution $v\in\bigcap_{1<p<\infty}\mathbf{L}^p(-1,1)$. This solution satisfies (5) and possesses a generalized derivative $v'\in\bigcup_{1<p<\infty}\mathbf{L}^p(-1,1)$, where both $v(x)$ and $v'(x)$ are bounded in a neighbourhood of $x=-1$ and belong to $\mathbf{C}^\infty(-1,1)$. Moreover, $\sqrt{1-x}\,v'(x)$ is locally Hölder continuous in each point of $(-1,1]$ with Hölder exponent $\frac{1}{2}$.*

The proof of the following lemma is given in the appendix Section 5.

Lemma 2.2. *For $x>0$, $y\geq 0$, and the function $\widetilde{k}_0(x,y)$ in (7), the repesentation*

$$\widetilde{k}_0(1+x,1+y)=\frac{1}{y+x}-\frac{6x}{(y+x)^2}+\frac{4x^2}{(y+x)^3}-h(x,y) \tag{9}$$

holds with a bounded and continuously differentiable function $h:(0,\infty)\times[0,\infty)\longrightarrow\mathbb{R}$.

Taking into account Lemma 2.1 and the formula ([12, Chapter II, Lemma 6.1])

$$\frac{d}{dt}\int_1^{1+2L}\frac{v_0(s)\,ds}{s-t}=\frac{v_0(1)}{1-t}-\frac{v_0(1+2L)}{1+2L-t}+\int_1^{1+2L}\frac{v_0'(s)\,ds}{s-t}\overset{(3)}{=}\frac{v_0(1)}{1-t}+\int_1^{1+2L}\frac{v_0'(s)\,ds}{s-t}\,,$$

we get

$$\begin{aligned}\frac{d}{dt}\int_1^{1+2L}\left[\frac{1}{s-t}+k_0(t,s)\right]v_0(s)\,ds\quad &\overset{(8)}{=}\quad \frac{d}{dt}\int_1^{1+2L}\frac{v_0(s)\,ds}{s-t}+\int_1^{1+2L}\frac{\partial\widetilde{k}_0(t,s)}{\partial s}v_0(s)\,ds\\ &=\quad \frac{v_0(1)}{1-t}-\widetilde{k}_0(t,1)v_0(1)+\int_1^{1+2L}\left[\frac{1}{s-t}-\widetilde{k}_0(t,s)\right]v_0'(s)\,ds\\ &\overset{(8)}{=}\quad \int_1^{1+2L}\left[\frac{1}{s-t}-\widetilde{k}_0(t,s)\right]v_0'(s)\,ds\,.\end{aligned}$$

Consequently, instead of (6) we can consider the integral equation

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{y-x}+\widetilde{k}_0^L(x,y)\right]u_0(y)\,dy=g_0^L(x),\quad -1<x<1,\tag{10}$$

where $g_0^L(x)=f_0'(1+L(1+x))$,

$$\widetilde{k}_0^L(x,y)=-L\,\widetilde{k}_0(1+L(1+x),1+L(1+y)),\quad\text{and}\quad u_0(x)=v_0'(1+L(1+x)).$$

Exploring (9), equation (10) takes the form

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{y-x}-\frac{1}{2+y+x}+\frac{6(1+x)}{(2+y+x)^2}-\frac{4(1+x)^2}{(2+y+x)^3}+h_0^L(x,y)\right]u_0(y)\,dy=g_0^L(x),\tag{11}$$

$-1<x<1$, where

$$h_0^L(x,y)=L\,h(L(1+x),L(1+y)).\tag{12}$$

We write (11) as

$$(\mathcal{A}_0+\mathcal{H}_0)u_0=g_0^L,\tag{13}$$

where

$$(\mathcal{A}_0u_0)(x)=\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{y-x}+\mathbf{h}_0\left(\frac{1+x}{1+y}\right)\frac{1}{1+y}\right]u_0(y)\,dy$$

with $\mathbf{h}_0(t)=-\dfrac{1}{1+t}+\dfrac{6t}{(1+t)^2}-\dfrac{4t^2}{(1+t)^3}$ and

$$(\mathcal{H}_0u_0)(x)=\frac{1}{\pi}\int_{-1}^{1}h_0^L(x,y)u_0(y)\,dy\,.$$

By using [2, Theorem 9.1], it was already mentioned in [1, Corollary 2.3] that the operator

$$\mathcal{A}_0:\mathbf{L}^2_\varphi\to\mathbf{L}^2_\varphi\tag{14}$$

is a bounded and invertible one, where $\varphi(x)=\sqrt{1-x^2}$. Here, for a Jacobi weight $\rho(x)=v^{\alpha,\beta}(x):=(1-x)^\alpha(1+x)^\beta$, by $\mathbf{L}^2_\rho$ there is denoted the Hilbert space of all (classes of) functions which are square integrable w.r.t. the weight $\rho(x)$, equipped with the inner product and the respective norm

$$\langle f,g\rangle_\rho:=\int_{-1}^{1}f(x)\overline{g(x)}\rho(x)\,dx\quad\text{and}\quad\|f\|_\rho:=\sqrt{\langle f,f\rangle_\rho}\,.$$

Let us denote the associated normalized orthogonal polynomial (with positive leading coefficient) of degree n by $p_n^\rho(x)$. Due to Lemma 2.1, the solution of (11) can be written in the form

$$u_0(x)=\frac{\widetilde{u}_0(x)}{\sqrt{1-x}}$$

with a bounded and locally Hölder continuous function $\widetilde{u}_0:(-1,1]\longrightarrow\mathbb{C}$, which is infinitely differentiable on $(-1,1)$. For that reason we are interested in approximate solutions of (11) of the form

$$\frac{p_n(x)}{\sqrt{1-x}}\,,\tag{15}$$

where $p_n(x)$ is an algebraic polynomial of degree less than n. The results on the stability of polynomial collocation methods [8, 6] (cf. also [9, 10, 4]) and collocation-quadrature methods [7] for Cauchy singular integral equations with additional fixed singularities of Mellin-type like in (11) do not cover the case (15), since in all these mentioned papers the approximate solution is of the form $(1-x)^\gamma(1+x)^\delta p_n(x)$ with $\gamma\neq0$ and $\delta\neq0$. Hence, we follow the approach described in [1, Section 2] and [5, Section 1] and use the isometrical isomorphism $\mathcal{J}_0:\mathbf{L}^2_\varphi\longrightarrow\mathbf{L}^2_\mu$, $f\mapsto\sqrt{1+.}f$, where again $\varphi(x)=\sqrt{1-x^2}$ and $\mu(x)=\sqrt{\dfrac{1-x}{1+x}}$, to get the following equation equivalent to (13),

$$(\mathcal{A}+\mathcal{H})u=g\tag{16}$$

with $\mathcal{A}=\mathcal{J}_0\mathcal{A}_0\mathcal{J}_0^{-1}:\mathbf{L}^2_\mu\longrightarrow\mathbf{L}^2_\mu$, $\mathcal{H}=\mathcal{J}_0\mathcal{H}_0\mathcal{J}_0^{-1}:\mathbf{L}^2_\mu\longrightarrow\mathbf{L}^2_\mu$, $g=\mathcal{J}_0g_0^L\in\mathbf{L}^2_\mu$, and $u=\mathcal{J}_0u_0$. It follows

$$(\mathcal{A}u)(x)=\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{y-x}+\mathbf{h}\left(\frac{1+x}{1+y}\right)\frac{1}{1+y}\right]u(y)\,dy$$

where $\mathbf{h}(t)=\sqrt{t}\,\mathbf{h}_0(t)-\dfrac{1}{1+\sqrt{t}}$, and

$$(\mathcal{H}u)(x)=\frac{1}{\pi}\int_{-1}^{1}h^L(x,y)u(y)\,dy\,,\quad h^L(x,y)=\sqrt{\frac{1+x}{1+y}}\,h_0^L(x,y),\quad g(x)=\sqrt{1+x}\,g_0^L(x).\tag{17}$$

In the following lemma we collect some mapping properties of the operators involved in equation (16). For this, as usual by $\mathbf{C}[-1,1]$ we denote the Banach space of all continuous functions $f:[-1,1]\longrightarrow\mathbb{C}$ equipped with the supremum norm $\|f\|_\infty$. Note that the operator $\mathcal{A}$ is the operator of an integral equation for the notched half-plane problem, for which a collocation-quadrature method (which we introduce in the following section) is studied in [5]. This enables us to use the results from [5] to study the properties of this numerical method applied to equation (16).

Lemma 2.3. *With the above notations the following holds.*

(a) *The operator* $\mathcal{A}:\mathbf{L}^2_\mu\longrightarrow\mathbf{L}^2_\mu$ *is bounded and invertible.*

(b) *The operators* $\mathcal{H}:\mathbf{L}^2_\mu\longrightarrow\mathbf{C}[-1,1]$ *and* $\mathcal{H}:\mathbf{L}^2_\mu\longrightarrow\mathbf{L}^2_\mu$ *are compact.*

Proof. Assertion (a) is a consequence of the already mentioned boundedness and invertibility of the operator (14). Since the function $[-1,1]^2\longrightarrow\mathbb{R},\,(x,y)\mapsto\sqrt{1+x}\,h_0^L(x,y)$ is continuous (see Lemma 2.2), the estimates

$$|(\mathcal{H}u)(x)|=\left|\int_{-1}^1\sqrt{1+x}\,h_0^L(x,y)\,\frac{u(y)\,dy}{\sqrt{1+y}}\right|\le\left\|\sqrt{1+x}\,h_0^L(x,.)\right\|_\infty\sqrt{\int_{-1}^1\frac{dy}{\sqrt{1-y^2}}}\,\|u\|_\mu$$

and

$$\begin{aligned}|(\mathcal{H}u)(x_1)-(\mathcal{H}u)(x_2)| &= \left|\int_{-1}^1\left[\sqrt{1+x_1}\,h_0^L(x_1,y)-\sqrt{1+x_2}\,h_0^L(x_2,y)\right]\frac{u(y)\,dy}{\sqrt{1+y}}\right| \\ &\le \left\|\sqrt{1+x_1}\,h_0^L(x_1,.)-\sqrt{1+x_2}\,h_0^L(x_2,.)\right\|_\infty\sqrt{\int_{-1}^1\frac{dy}{\sqrt{1-y^2}}}\,\|u\|_\mu\end{aligned}$$

show that the set $\left\{\mathcal{H}u:u\in\mathbf{L}^2_\mu,\,\|u\|_\mu\le1\right\}$ is a uniformly bounded and equicontinuous set of functions. Hence, $\mathcal{H}:\mathbf{L}^2_\mu\longrightarrow\mathbf{C}[-1,1]$ is compact together with $\mathcal{H}:\mathbf{L}^2_\mu\longrightarrow\mathbf{L}^2_\mu$ in virtue of the continuous imbedding $\mathbf{C}[-1,1]\subset\mathbf{L}^2_\mu$. ◻

3 A collocation-quadrature method

Here, we describe a collocation-quadrature method for the operator equation (16). Let $x^\sigma_{kn}=\cos\dfrac{2k-1}{2n}\pi$, $k=1,\dots,n$, $n\in\mathbb{N}$ denote the Chebyshev nodes of first kind and

$$\left(\mathcal{L}^\sigma_n f\right)(x)=\sum_{k=1}^n f(x^\sigma_{kn})\ell^\sigma_{kn}(x)$$

the respective interpolation polynomial of a function $f:(-1,1)\longrightarrow\mathbb{C}$, where $\ell^\sigma_{kn}(x)$ are the usual fundamental Lagrange interpolation polynomials w.r.t. these nodes. Moreover, let $\mathcal{M}^\sigma_n=\nu\mathcal{L}^\sigma_n\mu\mathcal{I}$ be the weighted interpolation operator given by

$$\left(\mathcal{M}^\sigma_n f\right)(x)=\nu(x)\left(\mathcal{L}^\sigma_n\mu f\right)(x)$$

with $\nu(x)=\sqrt{\dfrac{1+x}{1-x}}$ the Chebyshev weight of third kind. Finally, let $\mathcal{L}_n:\mathbf{L}^2_\mu\longrightarrow\mathbf{L}^2_\mu$ denote the orthogonal projection

$$\mathcal{L}_n f=\sum_{j=0}^{n-1}\left\langle f,\widetilde{p}_j\right\rangle_\mu\widetilde{p}_j\,,$$

where $\widetilde{p}_n=\nu p^\nu_n$, $n=0,1,\dots$ forms a complete orthonomal system in $\mathbf{L}^2_\mu$. In its image space $\operatorname{im}\mathcal{L}_n$ we look for an approximate solution u_n by solving

$$(\mathcal{A}_n+\mathcal{H}_n)u_n=\mathcal{M}^\sigma_n g\,,\tag{18}$$

where the operators $\mathcal{A}_n=\mathcal{M}^\sigma_n\left(\mathcal{S}+\mathcal{B}^0_n\right)\mathcal{L}_n$ and $\mathcal{H}_n=\mathcal{M}^\sigma_n\mathcal{H}^0_n\mathcal{L}_n$ are defined by

$$\begin{aligned}(\mathcal{S}u_n)(x) &= \frac{1}{\pi}\int_{-1}^1\frac{u_n(y)\,dy}{y-x}\,,\\ \left(\mathcal{B}^0_n u_n\right)(x) &= \frac{1}{\pi}\int_{-1}^1\left[\mathcal{L}^\sigma_n\varphi\mathbf{h}\left(\frac{1+x}{1+.}\right)\frac{u_n}{1+.}\right](y)\,\sigma(y)\,dy\,,\\ \left(\mathcal{H}^0_n u_n\right)(x) &= \frac{1}{\pi}\int_{-1}^1\left[\mathcal{L}^\sigma_n\varphi h^L(x,.)u_n\right](y)\,\sigma(y)\,dy\,,\end{aligned}$$

and $\sigma(x)=\dfrac{1}{\sqrt{1-x^2}}$. Note that, for the quadrature operators we have, for example,

$$\left(\mathcal{H}_n^0 u_n\right)(x)=\frac{1}{n}\sum_{k=1}^{n}\varphi(x_{kn}^\sigma)h^L(x,x_{kn}^\sigma)u_n(x_{kn}^\sigma)\,.$$

It is well known that, in the investigation of numerical methods for operator equations, the stability of the respective operator sequence plays an essential role. The sequence $(\mathcal{A}_n+\mathcal{H}_n)$ in (18) is called stable (in $\mathbf{L}^2_\mu$) if, for all sufficiently large n, the operators $\mathcal{A}_n+\mathcal{H}_n:\operatorname{im}\mathcal{L}_n\longrightarrow\operatorname{im}\mathcal{L}_n$ are invertible and if the norms $\left\|(\mathcal{A}_n-\mathcal{H}_n)^{-1}\mathcal{L}_n\right\|_{\mathcal{L}(\mathbf{L}^2_\mu)}$ are uniformly bounded. Note that, if the method is stable and if $(\mathcal{A}_n+\mathcal{H}_n)\mathcal{L}_n$ converges strongly to $\mathcal{A}+\mathcal{H}\in\mathcal{L}(\mathbf{L}^2_\mu)$, then the operator $\mathcal{A}+\mathcal{H}$ is injective. If additionally the image of $\mathcal{A}+\mathcal{H}$ equals $\mathbf{L}^2_\mu$, then $\left\|g-\mathcal{M}_n^\sigma g\right\|_\mu\longrightarrow 0$ implies the $\mathbf{L}^2_\mu$-convergence of the solution u_n of (18) to the (unique) solution $u\in\mathbf{L}^2_\mu$ of (16).

To investigate the stability of the operator sequence in (18) we follow the C^*-algebra approach already used in, for example, [4, 6, 7] (cf. also [8, 9, 10]). Moreover, in [5] a collocation-quadrature method for the numerical solution of an integral equation for the notched half plane problem was studied with the help of this approach. We will consider the operator sequence under consideration as an element of a C^*-algebra, which we describe in the following. For this, by ℓ^2 we denote the Hilbert space of all square summable sequences $\xi=\left(\xi_j\right)_{j=0}^\infty$, $\xi_j\in\mathbb{C}$ with the inner product $\langle\xi,\eta\rangle=\sum_{j=0}^\infty\xi_j\overline{\eta_j}$. Moreover, we define the operators

$$\mathcal{W}_n:\mathbf{L}^2_\mu\longrightarrow\mathbf{L}^2_\mu\,,\quad u\mapsto\sum_{j=0}^{n-1}\left\langle u,\widetilde{p}_{n-1-j}\right\rangle_\mu\widetilde{p}_j\,,\qquad \mathcal{P}_n:\ell^2\longrightarrow\ell^2\,,\quad (\xi_j)_{j=0}^\infty\mapsto(\xi_0,\cdots,\xi_{n-1},0,\dots)\,,$$

and

$$\mathcal{V}_n:\operatorname{im}\mathcal{L}_n\longrightarrow\operatorname{im}\mathcal{P}_n\,,\quad u_n\mapsto\left(\frac{\pi}{n}\sqrt{1-x_{1n}^\sigma}\,u_n(x_{1n}^\sigma),\dots,\frac{\pi}{n}\sqrt{1-x_{nn}^\sigma}\,u_n(x_{nn}^\sigma),0,\dots\right),$$
$$\widetilde{\mathcal{V}}_n:\operatorname{im}\mathcal{L}_n\longrightarrow\operatorname{im}\mathcal{P}_n\,,\quad u_n\mapsto\left(\frac{\pi}{n}\sqrt{1-x_{nn}^\sigma}\,u(x_{nn}^\sigma),\dots,\frac{\pi}{n}\sqrt{1-x_{1n}^\sigma}\,u(x_{1n}^\sigma),0,\dots\right).$$

Let $T=\{1,2,3,4\}$, set

$$\mathbf{X}^{(1)}=\mathbf{X}^{(2)}=\mathbf{L}^2_\mu,\quad \mathbf{X}^{(3)}=\mathbf{X}^{(4)}=\ell^2,\quad \mathcal{L}_n^{(1)}=\mathcal{L}_n^{(2)}=\mathcal{L}_n,\quad \mathcal{L}_n^{(3)}=\mathcal{L}_n^{(4)}=\mathcal{P}_n,$$

and define $\mathcal{E}_n^{(t)}:\operatorname{im}\mathcal{L}_n\longrightarrow\mathbf{X}_n^{(t)}:=\operatorname{im}\mathcal{L}_n^{(t)}$ for $t\in T$ by

$$\mathcal{E}_n^{(1)}=\mathcal{L}_n,\qquad \mathcal{E}_n^{(2)}=\mathcal{W}_n,\qquad \mathcal{E}_n^{(3)}=\mathcal{V}_n^\sigma,\qquad \mathcal{E}_n^{(4)}=\widetilde{\mathcal{V}}_n^\sigma.$$

Here and at other places, we use the notion $\mathcal{L}_n$, $\mathcal{W}_n$, … instead of $\mathcal{L}_n|_{\operatorname{im}\mathcal{L}_n}$, $\mathcal{W}_n|_{\operatorname{im}\mathcal{L}_n}$, … , respectively. All operators $\mathcal{E}_n^{(t)}$, $t\in T$ are unitary with the inverses

$$\left(\mathcal{E}_n^{(1)}\right)^{-1}=\mathcal{E}_n^{(1)},\quad \left(\mathcal{E}_n^{(2)}\right)^{-1}=\mathcal{E}_n^{(2)},\quad \left(\mathcal{E}_n^{(3)}\right)^{-1}=\mathcal{V}_n^{-1},\quad \left(\mathcal{E}_n^{(4)}\right)^{-1}=\widetilde{\mathcal{V}}_n^{-1},$$

where, for $\xi\in\operatorname{im}\mathcal{P}_n$,

$$\mathcal{V}_n^{-1}\xi=\sum_{k=1}^n\frac{n\,\xi_{k-1}}{\pi\sqrt{1-x_{kn}^\sigma}}\,\widetilde{\ell}_{kn}^\sigma\quad\text{and}\quad \widetilde{\mathcal{V}}_n^{-1}\xi=\sum_{k=1}^n\frac{n\,\xi_{n-k}}{\pi\sqrt{1-x_{kn}^\sigma}}\,\widetilde{\ell}_{kn}^\sigma,$$

and where

$$\widetilde{\ell}_{kn}^\sigma(x)=\frac{\nu(x)}{\nu(x_{kn}^\tau)}\,\ell_{kn}^\sigma(x),\quad k=1,\dots,n,$$

are the weighted fundamental interpolation polynomials. It is easily seen that, for all indices $r,t\in T$ with $r\neq t$, the operators

$$\mathcal{E}_n^{(r)}\left(\mathcal{E}_n^{(t)}\right)^{-1}\mathcal{L}_n^{(t)} \tag{19}$$

as well as their adjoints converge weakly to zero (cf., for example, the proof of [4, Lemma 2.1]). Now we can introduce the algebra of operator sequences we are interested in. By $\mathfrak{F}$ we denote the set of all sequences $(\mathcal{A}_n)$ of linear operators $\mathcal{A}_n:\operatorname{im}\mathcal{L}_n\longrightarrow\operatorname{im}\mathcal{L}_n$ for which the strong limits

$$\mathcal{W}^t(\mathcal{A}_n):=\lim_{n\to\infty}\mathcal{E}_n^{(t)}\mathcal{A}_n\left(\mathcal{E}_n^{(t)}\right)^{-1}\mathcal{L}_n^{(t)}$$

and

$$(\mathcal{W}^t(\mathcal{A}_n))^*=\lim_{n\to\infty}\left(\mathcal{E}_n^{(t)}\mathcal{A}_n\left(\mathcal{E}_n^{(t)}\right)^{-1}\mathcal{L}_n^{(t)}\right)^*,\quad t\in T\,,$$

exist. If $\mathfrak{F}$ is provided with the supremum norm $\|(\mathcal{A}_n)\|_{\mathfrak{F}}:=\sup_{n\ge1}\|\mathcal{A}_n\mathcal{L}_n\|_{\mathfrak{L}(\mathbf{L}^2_\nu)}$ and with the operations $(\mathcal{A}_n)+(\mathcal{B}_n):=(\mathcal{A}_n+\mathcal{B}_n)$, $(\mathcal{A}_n)(\mathcal{B}_n):=(\mathcal{A}_n\mathcal{B}_n)$, and $(\mathcal{A}_n)^*:=(\mathcal{A}_n^*)$, then $\mathfrak{F}$ becomes a C^*-algebra with the identity element $(\mathcal{L}_n)$. Furthermore, we introduce the set $\mathfrak{J}\subset\mathfrak{F}$ of all sequences of the form

$$\left(\sum_{t=1}^4\left(\mathcal{E}_n^{(t)}\right)^{-1}\mathcal{L}_n^{(t)}\mathcal{T}_t\mathcal{E}_n^{(t)}+\mathcal{C}_n\right),$$

where the linear operators $\mathcal{T}_t:\mathbf{X}^{(t)}\longrightarrow\mathbf{X}^{(t)}$ are compact and where the sequence $(\mathcal{C}_n)\in\mathfrak{F}$ belongs to the closed ideal $\mathfrak{G}$ of all sequences from $\mathfrak{F}$ tending to zero in norm, i.e., $\lim_{n\to\infty}\|\mathcal{C}_n\mathcal{L}_n\|_{\mathfrak{L}(\mathbf{L}^2_\nu)}=0$. From [13, 14, Theorem 10.33] (see also [3, Theorem 6.1]) we infer the following proposition.

Proposition 3.1. *The set $\mathfrak{J}$ forms a two-sided closed ideal in the C^*-algebra $\mathfrak{F}$. Moreover, a sequence $(\mathcal{A}_n) \in \mathfrak{F}$ is stable if and only if the operators $\mathcal{W}^t(\mathcal{A}_n) : \mathbf{X}^{(t)} \longrightarrow \mathbf{X}^{(t)}$, $t \in T$ and the coset $(\mathcal{A}_n) + \mathfrak{J} \in \mathfrak{F}/\mathfrak{J}$ are invertible.*

Let $\mathfrak{A}_0$ denote the smallest C^*-subalgebra of $\mathfrak{F}$ containing all sequences from $\mathfrak{J}$ and all sequences $(\mathcal{A}_n)$ with

$$\mathcal{A}_n = \mathcal{M}_n^\sigma(a\mathcal{I} + b\mathcal{S} + \mathcal{B}_n^0)\mathcal{L}_n ,$$

where a, b are piecewise continuous functions on $[-1,1]$, where $\mathcal{I} : \mathbf{L}^2_\mu \longrightarrow \mathbf{L}^2_\mu$ is the identity operator, and where $\mathcal{S}$ as well as $\mathcal{B}_n^0$ are defined after (18). Then, the following is proved in [5, Theorem 3.12].

Proposition 3.2. *A sequence $(\mathcal{A}_n) \in \mathfrak{A}_0$ is stable in $\mathbf{L}^2_\mu$ if and only if all limit operators $\mathcal{W}^t(\mathcal{A}_n) : \mathbf{X}^{(t)} \longrightarrow \mathbf{X}^{(t)}$, $t = 1,2,3,4$ are invertible.*

The following statement we infer from [5, Section 4].

Lemma 3.3. *Let $\mathcal{A}_n$ be the operators in (18), i.e., $\mathcal{A}_n = \mathcal{M}_n^\sigma(\mathcal{S} + \mathcal{B}_n^0)\mathcal{L}_n$. Then, the limit operators $\mathcal{W}^1(\mathcal{A}_n), \mathcal{W}^2(\mathcal{A}_n) : \mathbf{L}^2_\mu \longrightarrow \mathbf{L}^2_\mu$ and $\mathcal{W}^3(\mathcal{A}_n) : \ell^2 \longrightarrow \ell^2$ are invertible, and the fourth limit operator $\mathcal{W}^4(\mathcal{A}_n) : \ell^2 \longrightarrow \ell^2$ is Fredholm with index 0.*

Now, we are turning to study the remaining part of the operator sequence involved in equation (18), namely $(\mathcal{H}_n)$. First, we remark that (see [4, Corollary 3.3])

$$\lim_{n\to\infty} \left\| f - \mathcal{L}_n^\sigma f \right\|_\nu = 0 \quad \text{for all} \quad f \in \mathbf{C}_{\gamma,\delta} , \tag{20}$$

where $0 \le \gamma < \frac{1}{4}$, $0 \le \delta < \frac{3}{4}$, where again $\nu = \mu^{-1}$, and where $\mathbf{C}_{\gamma,\delta}$ denotes the Banach space consisting of all continuous functions $f : (-1,1) \longrightarrow \mathbb{C}$ for which $v^{\gamma,\delta} f : [-1,1] \longrightarrow \mathbb{C}$ is continuous with $\left(v^{\gamma,\delta} f\right)(1) = 0$ if $\gamma > 0$ and $\left(v^{\gamma,\delta} f\right)(-1) = 0$ if $\delta > 0$. The norm in $\mathbf{C}_{\gamma,\delta}$ is defined by

$$\|f\|_{\gamma,\delta,\infty} := \left\| v^{\gamma,\delta} f \right\|_\infty .$$

Furthermore (see [5, Corollary 2.12]),

$$\lim_{n\to\infty} \left\| f - \mathcal{M}_n^\sigma f \right\|_\mu = 0 \quad \text{for all} \quad f \in \mathbf{C}_{\alpha,\beta} , \tag{21}$$

where $0 \le \alpha < \frac{3}{4}$ and $0 \le \beta < \frac{1}{4}$.

Lemma 3.4. *Assume that the function $[-1,1]^2 \longrightarrow \mathbb{C}$, $(x,y) \mapsto r(x,y)v^{\alpha,\beta}(y)$ is continuous, where $0 \le \alpha < \frac{1}{4}$ and $0 \le \beta < \frac{3}{4}$. Then,*

$$\lim_{n\to\infty} \sup\left\{ \left\| \mathcal{L}_n^\sigma r(x,.) - r(x,.) \right\|_\nu : -1 \le x \le 1 \right\} = 0 .$$

Proof. Fix $\gamma, \delta \in \mathbb{R}$ such that $\alpha < \gamma < \frac{1}{4}$ and $\beta < \delta < \frac{3}{4}$. By assumption $r^x \in \mathbf{C}_{\gamma,\delta}$, where $r^x(y) = r(x,y)$, and

$$\lim_{x\to x_0} \|r^x - r^{x_0}\|_{\gamma,\delta,\infty} = 0 \quad \text{for all} \quad x_0 \in [-1,1] .$$

Suppose that the assertion of the lemma is not true. Then, there is an $\varepsilon > 0$ and a sequence $n_1 < n_2 < \ldots$ of natural numbers satisfying

$$\sup\left\{ \left\| \mathcal{L}_{n_k}^\sigma r^x - r^x \right\|_\nu : -1 \le x \le 1 \right\} \ge 2\varepsilon \quad \text{for all} \quad k \in \mathbb{N} .$$

Hence, for every $k \in \mathbb{N}$, there is an $x_k \in [-1,1]$ such that $\left\| \mathcal{L}_{n_k}^\sigma r^{x_k} - r^{x_k} \right\|_\nu \ge \varepsilon$, and we can assume that $x_k \longrightarrow x^*$ if $k \longrightarrow \infty$. By (20), $M := \sup\left\{ \left\| \mathcal{L}_n^\sigma \right\|_{\mathbf{C}_{\gamma,\delta}\to\mathbf{L}^2_\nu} : n \in \mathbb{N} \right\} < \infty$ and

$$M_0 := \sqrt{\int_{-1}^1 (1-x)^{-\frac{1}{2}-2\gamma}(1+x)^{\frac{1}{2}-2\delta}\, dx} < \infty .$$

Moreover, there is a $k_0 \in \mathbb{N}$ such that

$$\left\| \mathcal{L}_{n_k}^\sigma r^{x^*} - r^{x^*} \right\|_\nu < \frac{\varepsilon}{3} \quad \text{and} \quad \left\| r^{x_k} - r^{x^*} \right\|_{\gamma,\delta,\infty} < \frac{\varepsilon}{3\min\{M_0, M\}} \quad \text{for all} \quad k \ge k_0 .$$

It follows, for $k \ge k_0$,

$$\begin{aligned} \varepsilon \le \left\| \mathcal{L}_{n_k}^\sigma r^{x_k} - r^{x_k} \right\|_\nu &\le \left\| \mathcal{L}_{n_k}^\sigma (r^{x_k} - r^{x^*}) \right\|_\nu + \left\| \mathcal{L}_{n_k}^\sigma r^{x^*} - r^{x^*} \right\|_\nu + \left\| r^{x^*} - r^{x_k} \right\|_\nu \\ &\le M \left\| r^{x_k} - r^{x^*} \right\|_{\gamma,\delta,\infty} + \frac{\varepsilon}{3} + M_0 \left\| r^{x_k} - r^{x^*} \right\|_{\gamma,\delta,\infty} < \varepsilon , \end{aligned}$$

which is a contradiction. $\square$

Lemma 3.5. *The sequence $(\mathcal{H}_n)$ belongs to the ideal $\mathfrak{J}$ of the algebra $\mathfrak{F}$. In particular, $\mathcal{W}^1(\mathcal{H}_n) = \mathcal{H}$ and $\mathcal{W}^t(\mathcal{H}_n) = 0$ for $t = 2,3,4$.*

Proof. Due to $\mathcal{L}_n \longrightarrow \mathcal{I}$ strongly in $\mathbf{L}^2_\mu$, due to (21), and due to Lemma 2.3,(b), the sequence $\left(\mathcal{L}_n \mathcal{H} \mathcal{L}_n - \mathcal{M}_n^\sigma \mathcal{H} \mathcal{L}_n\right)$ belongs to the ideal $\mathfrak{G}$. Since, for $u_n = v p_n \in \operatorname{im} \mathcal{L}_n$, $\varphi u_n = (1+.)p_n$ is a polynomial of degree at most n, we have

$$\left(\mathcal{H}_n^0 u_n\right)(x) = \frac{1}{\pi}\int_{-1}^{1} \left[\mathcal{L}_n^\sigma h^L(x,.)\right](y) p_n(y) v(y)\, dy$$

and consequently, again using (21),

$$\begin{aligned} \left\|\mathcal{M}_n^\sigma \mathcal{H}\mathcal{L}_n u_n - \mathcal{H}_n u_n\right\|_\mu &\le \left\|\mathcal{M}_n^\sigma\right\|_{\mathbf{C}\to\mathbf{L}^2_\mu} \left\|\mathcal{H}u_n - \mathcal{H}_n^0 u_n\right\|_\infty \\ &\le c \sup\left\{\int_{-1}^{1} \left|h^L(x,y) - \left[\mathcal{L}_n^\sigma h^L(x,.)\right](y)\right| |p_n(y)| v(y)\, dy : -1 \le x \le 1\right\} \\ &\le c \sup\left\{\left\|\left[h^L(x,.) - \mathcal{L}_n^\sigma h^L(x,.)\right]\right\|_v : -1 \le x \le 1\right\} \|p_n\|_v \le c_n \|u_n\|_\mu , \end{aligned}$$

where $\lim_{n\to\infty} c_n = 0$ in virtue of the continuity of $h^L(x,y)\sqrt{1+y}$ on $[-1,1]^2$ (cf. Lemma 2.2, (12), and (17)) and Lemma 3.4. Hence, also $\left(\mathcal{M}_n^\sigma \mathcal{H}\mathcal{L}_n - \mathcal{H}_n\right)$ belongs to the ideal $\mathfrak{G}$. Now, the assertion follows by using the compactness of $\mathcal{H}$, the strong convergence of $\mathcal{L}_n$, and the weak convergences of the operators (19) and their adjoints implying $(\mathcal{L}_n \mathcal{H}\mathcal{L}_n) \in \mathfrak{J}$ together with $\mathcal{W}^1(\mathcal{H}_n) = \mathcal{H}$ and $\mathcal{W}^t(\mathcal{H}_n) = 0$ for $t = 2,3,4$. ◻

Combining Proposition 3.2, Lemma 3.3, and Lemma 3.5, we get the following stability result.

Proposition 3.6. *The sequence $(\mathcal{A}_n + \mathcal{H}_n)$ of the operators in the collocation-quadrature method (18) is stable in $\mathbf{L}^2_\mu$, if and only if the homogeneous equation $(\mathcal{A}+\mathcal{H})u = 0$ has only the trivial solution in $\mathbf{L}^2_\mu$ and if also the operator*

$$-\mathbf{S} + \mathbf{H} : \ell^2 \longrightarrow \ell^2 \tag{22}$$

has a trivial null space, where

$$\mathbf{S} = \left[\ \frac{1-(-1)^{j-k}}{j-k} - \frac{1-(-1)^{j+k+1}}{j+k+1}\ \right]_{j,k=0}^{\infty} \quad \textit{and} \quad \mathbf{H} = \left[\ \mathbf{h}\left(\frac{(j+\frac12)^2}{(k+\frac12)^2}\right)\frac{2}{k+\frac12}\ \right]_{j,k=0}^{\infty}.$$

Proof. Due to Proposition 3.2, the stability of the operator sequence $(\mathcal{A}_n + \mathcal{H}_n)$ is equivalent to the invertibility of all limit operators $\mathcal{W}^t(\mathcal{A}_n + \mathcal{H}_n)$. Since the first operator equals $\mathcal{W}^1(\mathcal{A}_n + \mathcal{H}_n) = \mathcal{A} + \mathcal{H}$ (see Lemma 3.5 and [5, Prop. 2.17]) and since Lemma 2.3 is in force, the invertibility of this operator is equivalent to the triviality of its null space. Moreover, by Lemma 3.3 and Lemma 3.5, the second and third limit operators are invertible. The fourth limit operator equals (see [5, Prop. 2.17]) $\pi^{-1}(-\mathbf{S}+\mathbf{H}) : \ell^2 \longrightarrow \ell^2$ and is Fredholm with index zero (see [5, Section 4]), and the proposition is proved. ◻

Remark 1. Here, we focus on the use of the Chebyshev nodes of first kind as collocation nodes, since it is obvious from the results of [5, Section 4] that the collocation-quadrature method based on the Chebyshev nodes of third kind is unstable. For Chebyshev nodes of second and fourth kind, we are not able to prove that the respective operator sequences belong to the corresponding algebra $\mathfrak{F}$.

4 Computational aspects and numerical results

Before presenting numerical results obtained by applying the method introduced in Section 3, let us discuss the method and results given in [11] for equation (1) resp. (6). Kalandiya [11, Section 37] applies directly a collocation-quadrature method to equation (6) after multiplying the unknown function by $\sqrt{1+y}$ and the equation by $\sqrt{1+x}$. Hence, the mentioned method is applied to (cf. [11, (37.8)])

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{y-x} + k_*(x,y)\right] v_*(y)\, dy = f_*(x) + C\sqrt{1+x}, \quad -1 < x < 1, \tag{23}$$

where

$$k_*(x,y) = \frac{1}{\sqrt{1+y}}\left[\sqrt{1+x}\, k_0^L(x,y) - \frac{1}{\sqrt{1+y}+\sqrt{1+x}}\right], \quad f_*(x) = \sqrt{1+x}\, f_0^L(x),$$

and $v_*(x) = \sqrt{1+x}\, v(x)$ with $v(x)$ and $f_0^L(x)$ from (6). An approximate solution $v_n(x)$ for $v_*(x)$ is searched for in the form (cf. [11, (37.9),(37.10)])

$$v_n(x) = \sum_{k=1}^{n} \xi_{kn} \widetilde{\ell}^{\varphi}_{kn}(x), \tag{24}$$

where $\widetilde{\ell}^{\varphi}_{kn}(x) = \dfrac{\varphi(x)\ell^{\varphi}_{kn}(x)}{\varphi(x^{\varphi}_{kn})}$ and where $\ell^{\varphi}_{kn}(x) = \dfrac{U_n(x)}{(x-x^{\varphi}_{kn})U_n'(x^{\varphi}_{kn})}$ with $U_n(\cos\theta) = \dfrac{\sin n\theta}{\sin s}$ are the fundamental Lagrange interpolation polynomials w.r.t. the Chebyshev nodes $x^{\varphi}_{kn} = \cos\dfrac{k\pi}{n+1}$, $k = 1,\dots,n$ of second kind. The integral operator with the kernel function $k_*(x,y)$ is approximated with the help of the Gaussian rule w.r.t. the x_{kn}^{φ}'s. After substituting $v_n(x)$ into (23) and collocating at x_{jn}^{φ}, $j=1,\ldots,n$ one obtains (by using relation (27) below) a system of linear equations

$$\sum_{k=1}^{n}\alpha_{jk}\xi_{kn}-\sqrt{1+x_{jn}^{\varphi}}\,C=f_*(x_{jn}^{\varphi}),\quad j=1,\ldots,n, \tag{25}$$

where (cf. [11, (37.11)] and also [9, (3.10)])

$$\alpha_{jk}=\left[\frac{1-(-1)^{j+k}}{x_{kn}^{\varphi}-x_{jn}^{\varphi}}+k_*(x_{jn}^{\varphi},x_{kn}^{\varphi})\right]\frac{\varphi(x_{kn}^{\varphi})}{n+1}\quad j,k=1,\ldots,n. \tag{26}$$

The first addend in the brackets equals 0 in case of $j=k$. If one considers equation (23) in the space $\mathbf{L}^2_\sigma$ with $\sigma(x)=(1-x^2)^{-\frac{1}{2}}$ again being the Chebyshev weight of first kind, then from the well-known relation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(y)U_n(y)}{y-x}\,dy=-T_{n+1}(x),\quad -1<x<1,\quad n=0,1,2,\ldots, \tag{27}$$

where the Chebyshev polynomials of second kind $U_n(x)=\sqrt{\frac{\pi}{2}}\,p_n^{\varphi}(x)$ are already mentioned after equation (24) and where $T_n(x)$ with $T_n(\cos\theta)=\sqrt{\frac{\pi}{2}}\,p_n^{\sigma}(\cos\theta)=\cos n\theta$, $n\ge 1$, are the Chebyshev polynomials of first kind, one can conclude the following: If $(v_*,C)\in\mathbf{L}^2_\sigma\times\mathbb{R}$ is a solution of (23) then

$$\int_{-1}^{1}\left[f_*(x)+C\sqrt{1+x}-\frac{1}{\pi}\int_{-1}^{1}k_*(x,y)v_*(y)\,dy\right]\sigma(x)\,dx=0. \tag{28}$$

Kalandiya uses this condition (cf. [11, (37.129,(37.13)]) to get a additional equation to the system (25) by discretizing (28) with the help of the Gaussian rule w.r.t. the Chebyshev nodes of first kind and with the help of the already mentioned discretization of the integral operator with the kernel function $k_*(x,y)$. Finally, Kalandiya ends up with a system

$$\mathbb{A}_n\xi_n=\eta_n \tag{29}$$

of linear equations, where $\xi_n=\left[\ \xi_{kn}\ \right]_{k=1}^{n+1}$ is the vector of the function values $\xi_{kn}=v_n(x_{kn}^{\varphi})$, $k=1,\ldots,n$, and the approximate value $\xi_{n+1,n}$ of C, where $\eta_n=\left[\ \eta_{jn}\ \right]_{j=1}^{n+1}$ with $\eta_{jn}=f_*(x_{jn}^{\varphi})$, $j=1,\ldots,n$, and $\eta_{n+1,n}=\frac{1}{n}\sum_{k=1}^{n}f_*(x_{kn}^{\sigma})$, and where the system matrix $\mathbb{A}_n=\left[\ \alpha_{jk}\ \right]_{j,k=1}^{n+1}$ is given by (26) and by

$$\alpha_{n+1,k}=\frac{1}{n(n+1)}\sum_{j=1}^{n}k_*(x_{jn}^{\sigma},x_{kn}^{\varphi})\varphi(x_{kn}^{\varphi}),\quad \alpha_{k,n+1}=-\sqrt{1+x_{kn}^{\varphi}},\quad k=1,\ldots,n,$$

as well as $\alpha_{n+1,n+1}=-\frac{2\sqrt{2}}{\pi}$. But, condition (28) is an artifical one, since one can only say that this condition is satisfied if $(v_*,C)\in\mathbf{L}^2_\sigma\times\mathbb{R}$ is a solution of (23). One cannot use it as a solvability condition for equation (23). Moreover, one should note that, due to the considerations by Duduchava [2, Section 14], the operator defined by the left hand side of (6) has Fredholm index 0 in the space $\mathbf{L}^2_\nu$, where $\nu(x)=v^{-\frac{1}{2},\frac{1}{2}}(x)$, which means that the operator given by the left hand side of (23) has Fredholm index 0 in $\mathbf{L}^2_\sigma$. These problems are also confirmed by the numerical results, which we present in Table 2 below. In particular, one is interested in the computation of a normalized stress intensity factor, which is independent of P (cf. (2)) and is given in case of $P=1$ by the formula (cf. [11, (36.35)])

$$\delta=\frac{1}{L}\lim_{x\to 1-0}\frac{v_*(x)}{\sqrt{1-x}}. \tag{30}$$

Using (24), δ is approximated by (cf. [11, (37.16),(36.35)])

$$\delta_n=\frac{\sqrt{2}}{L}\sum_{k=1}^{n}\frac{\xi_{kn}\ell_{kn}^{\varphi}(1)}{\varphi(x_{kn}^{\varphi})}=\frac{\sqrt{2}}{L}\sum_{k=1}^{n}(-1)^{k+1}\xi_{kn}\sqrt{\frac{1+x_{kn}^{\varphi}}{1-x_{kn}^{\varphi}}}=\frac{\sqrt{2}}{L}\sum_{k=1}^{n}(-1)^{k+1}\xi_{kn}\cot\frac{k\pi}{2(n+1)}. \tag{31}$$

Kalandiya [11, p. 257] presents the following results:

L	10.0	5.0	1.0	0.2	0.04	0.01
δ	1.1338	1.2006	1.6281	2.9460	4.3970	4.9063

Table 1: [11, p. 257]

From Table 2 we observe that the results of Table 1 are obviously obtained for $n=40$ and that the computed approximate values for δ strongly depend on n, which indicates instability of the method (cf. also Table 3).

L \ n	40	80	160	320	640	1280
0.01	4.9064	4.8802	4.8571	4.8369	4.8192	4.8036
0.04	4.3970	4.3714	4.3486	4.3287	4.3113	4.2960
0.20	2.9461	2.9237	2.9037	2.8862	2.8709	2.8574
1.00	1.6281	1.6130	1.5994	1.5876	1.5773	1.5683
5.00	1.2006	1.1908	1.1816	1.1737	1.1669	1.1610
10.00	1.1338	1.1253	1.1172	1.1102	1.1042	1.0991

Table 2: δ_n from (31) obtained by KALANDIYA's method (29)

Since the $\mathbf{L}^2_\sigma$-norm of $v_n = \varphi\, p_n$ is equal to

$$\sqrt{\int_{-1}^{1} \sqrt{1-x^2}\, |p_n(x)|^2\, dx} = \sqrt{\frac{\pi}{n+1}\sum_{k=1}^{n}\left[\varphi(x_{kn}^{\varphi})\right]^2 \left|p_n(x_{kn}^{\varphi})\right|^2} = \sqrt{\frac{\pi}{n+1}\sum_{k=1}^{n}|\xi_{kn}|^2}\,,$$

the condition numbers of the matices $\mathbb{A}_n$ in (29) should be bounded if the collocation-quadrature method, represented by (29), is stable in $\mathbf{L}^2_\sigma$. Unfortunately, the results shown in Table 3 imply that this is not the case.

L \ n	40	80	160	320	640	1280
0.01	2.06E04	1.42E05	1.02E06	7.47E06	5.55E07	4.15E08
0.04	1.94E04	1.34E05	9.62E05	7.05E06	5.23E07	3.91E08
0.20	1.58E04	1.09E05	7.76E05	5.67E06	4.20E07	3.14E08
1.00	1.05E04	7.21E04	5.13E05	3.73E06	2.76E07	2.05E08
5.00	6.36E03	4.36E04	3.09E05	2.24E06	1.64E07	1.22E08
10.00	4.96E03	3.41E04	2.42E05	1.75E06	1.18E07	9.48E07

Table 3: $\operatorname{cond}(\mathbb{A}_n)$ for $\mathbb{A}_n$ from (29)

In the Tables 4 and 5 we present the numerical results obtained by applying the collocation-quadrature method described in Section 3 to equation (16), i.e., to the equation (cf. (10), (16), and (17))

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{y-x} + \widetilde{k}^L(x,y)\right]u(y)\,dy = g(x), \quad -1<x<1, \tag{32}$$

with

$$\widetilde{k}^L(x,y) = \frac{1}{\sqrt{1+y}}\left[\sqrt{1+x}\,\widetilde{k}_0^L(x,y) - \frac{1}{\sqrt{1+y}+\sqrt{1+x}}\right], \quad g(x) = \sqrt{1+x}\, g_0^L(x),$$

and $u(x) = \sqrt{1+x}\, u_0(x)$. The respective system

$$\mathbb{B}_n \xi_n = \eta_n \tag{33}$$

of linear equations with the system matrix $\mathbb{B}_n = \mathbb{S}_n + \mathbb{K}_n$ and the right hand side η_n is given by

$$\begin{aligned}
\mathbb{S}_n &= \left[\sqrt{\frac{1-x_{jn}^\sigma}{1-x_{kn}^\sigma}}\,\big(\mathcal{S}\widetilde{\ell}_{kn}^\sigma\big)(x_{jn}^\sigma)\right]_{j,k=1}^{n}, \\
\mathbb{K}_n &= \left[\sqrt{\frac{1-x_{jn}^\sigma}{1-x_{kn}^\sigma}}\,\big((\mathcal{B}_n^0+\mathcal{H}_n^0)\widetilde{\ell}_{kn}^\sigma\big)(x_{jn}^\sigma)\right]_{j,k=1}^{n} = \left[\frac{1}{n}\sqrt{\frac{1-x_{jn}^\sigma}{1-x_{kn}^\sigma}}\,\varphi(x_{kn}^\sigma)\widetilde{k}^L(x_{jn}^\sigma,x_{kn}^\sigma)\right]_{j,k=1}^{n} \\
&= \left[\frac{1}{n}\sqrt{1-x_{jn}^\sigma}\sqrt{1+x_{kn}^\sigma}\,\widetilde{k}^L(x_{jn}^\sigma,x_{kn}^\sigma)\right]_{j,k=1}^{n},
\end{aligned}$$

and $\eta_n = \left[\sqrt{1-x_{jn}^\sigma}\, g(x_{jn}^\sigma)\right]_{j=1}^{n}$. To use, for example, the matrix $\mathbb{S}_n$ instead of the matrix $\mathbb{S}_n^0 = \left[\big(\mathcal{S}\widetilde{\ell}_{kn}^\sigma\big)(x_{jn}^\sigma)\right]_{j,k=1}^{n}$ is motivated by the following fact. The matrix $\mathbb{S}_n + \mathbb{K}_n$ is equal to the operator $\mathcal{E}_n^{(3)}(\mathcal{A}_n+\mathcal{H}_n)\big(\mathcal{E}_n^{(3)}\big)^{-1} : \operatorname{im}\mathcal{P}_n \longrightarrow \operatorname{im}\mathcal{P}_n$. Hence, since $\mathcal{E}_n^{(3)} : \operatorname{im}\mathcal{L}_n \longrightarrow \operatorname{im}\mathcal{P}_n$ is a unitary operator, the sequence of the matrices $\mathbb{S}_n + \mathbb{K}_n$ is stable if and only if the sequence of the operators $\mathcal{A}_n + \mathcal{H}_n$ in (18) is stable in $\mathbf{L}^2_\sigma$. This means, that in case of stability the matrix $\mathbb{S}_n + \mathbb{K}_n$ is a preconditioning of the matrix $\mathbb{S}_n^0 + \mathbb{K}_n^0$.

To compute δ in (30) in terms of the solution $u(x)$ of equation (32) we proceed as follows. By definition of $v_*(x)$ we have, with $v(x)$ from (6) and $u(x)$ from (32),

$$\delta = \frac{\sqrt{2}}{L}\lim_{x\to 1-0}\frac{v(x)}{\sqrt{1-x}} = -\frac{2\sqrt{2}}{L}\lim_{x\to 1-0} v'(x)\sqrt{1-x} = -2\lim_{x\to 1-0} u(x)\sqrt{1-x}\,, \tag{34}$$

since $v'(x) = L\,v_0'(1+L(1+x)) = L\,u_0(x) = \dfrac{L\,u(x)}{\sqrt{1+x}}$ with $v_0(x)$ from (1) and $u_0(x)$ from (10). Hence, with the help of the solution $\xi_n = \left[\ \xi_{kn}\ \right]_{k=1}^n = \left[\ \sqrt{1-x_{kn}^\sigma}\,u_n(x_{kn}^\sigma)\ \right]_{k=1}^n$ and the representation $u_n(x) = \sqrt{\dfrac{1+x}{1-x}}\,p_n(x)$, we can approximate δ by

$$\delta_n = \sqrt{2}\,p_n(1) = -2\sqrt{2}\sum_{k=1}^n p_n(x_{kn}^\sigma)\ell_{kn}^\sigma(1) = -\frac{2\sqrt{2}}{n}\sum_{k=1}^n(-1)^{k+1}p_n(x_{kn}^\sigma)\sqrt{\frac{1+x_{kn}^\sigma}{1-x_{kn}^\sigma}} = \frac{2\sqrt{2}}{n}\sum_{k=1}^n(-1)^k\frac{\xi_{kn}}{\sqrt{1-x_{kn}^\sigma}}\,. \tag{35}$$

L \ n	40	80	160	320	640	1280
0.01	4.5604	4.5607	4.5607	4.5607	4.5607	4.5608
0.04	4.0615	4.0618	4.0618	4.0618	4.0618	4.0618
0.20	2.6663	2.6664	2.6665	2.6665	2.6665	2.6665
1.00	1.4575	1.4576	1.4576	1.4576	1.4576	1.4576
5.00	1.1021	1.1021	1.1021	1.1021	1.1021	1.1021
10.00	1.0527	1.0527	1.0527	1.0527	1.0527	1.0527

Table 4: δ_n from (35)

L \ n	40	80	160	320	640	1280
0.01	2.84	3.04	3.20	3.33	3.45	3.55
0.04	2.78	2.99	3.16	3.30	3.42	3.52
0.20	2.60	2.83	3.03	3.19	3.32	3.44
1.00	2.56	2.88	3.15	3.37	3.56	3.71
5.00	2.81	3.32	3.76	4.12	4.43	4.68
10.00	2.96	3.59	4.13	4.60	4.98	5.30

Table 5: cond($\mathbb{B}_n$) for $\mathbb{B}_n$ from (33)

Conclusion. From Table 5 we can conclude that the condition on the null space of the operator $\mathcal{A}+\mathcal{H}$ and condition (22) in Proposition 3.6 seem to be satisfied and the collocation-quadrature method (18) is stable in $\mathbf{L}^2_\mu$. The approximate values for δ given in Table 4 differ from the values given in Tables 1 and 2, but do almost not depend on n. Hence, one can assume that the values of δ given in Table 4 are more realistic than the values given in Tables 1 and 2.

In [11, p. 257] it is claimed that $\delta \searrow 1$ for $L \nearrow \infty$. This is confirmed by the values presented in Table 6, which are obtained by (33) and (35) for $n = 500$.

L	10.0	100.0	1000.0	10000.0
δ	1.0527	1.0055	1.0006	1.0001

Table 6: δ_n from (35) for $n = 500$

5 Appendix: Proof of Lemma 2.2

By (7) we have

$$\begin{aligned}\widetilde{k}_0(1+x,1+y) &= \frac{x+1}{y+x+xy} - \left[x+1+\frac{1}{x+1}-\frac{2}{(x+1)^3}\right]\frac{1}{(y+x+xy)^2}\\ &\quad+\left[x+1-\frac{2}{x+1}+\frac{1}{(x+1)^3}\right]\frac{1}{(y+x+xy)^3}\\ &\quad-\left[1+\frac{1}{(x+1)^2}\right]\frac{1}{y+1}-\frac{1}{x+1}-\frac{1}{(x+1)^3}\,.\end{aligned}$$

If we take into account that

$$\begin{aligned}x+1+\frac{1}{x+1}-\frac{2}{(x+1)^3} &= 6x+\frac{(x+1)^3+(x+1)^2-2-5x(x+1)^3}{(x+1)^3}\\ &= 6x+\frac{x^3+3x^2+3x+1+x^2+2x+1-2-5x^4-15x^3-15x^2-5x}{(x+1)^3}\\ &= 6x-\frac{x^2(5x^2+14x+11)}{(x+1)^3}\end{aligned}$$

and

$$\begin{aligned} x+1-\frac{2}{x+1}+\frac{1}{(x+1)^3} &= 4x^2+\frac{(x+1)^4-2(x+1)^2+1-4x^2(x+1)^3}{(x+1)^3} \\ &= 4x^2+\frac{x^4+4x^3+6x^2+4x+1-2x^2-4x-2+1-4x^5-12x^4-12x^3-4x^2}{(x+1)^3} \\ &= 4x^2-\frac{x^3(4x^2+11x+8)}{(x+1)^3} \end{aligned}$$

hold, we can compute

$$\begin{aligned} \widetilde{k}_0(1+x,1+y) &= \frac{x+1}{y+x+xy}-\left[x+1+\frac{1}{x+1}-\frac{2}{(x+1)^3}\right]\frac{1}{(y+x+xy)^2} \\ &\quad +\left[x+1-\frac{2}{x+1}+\frac{1}{(x+1)^3}\right]\frac{1}{(y+x+xy)^3}-\left[1+\frac{1}{(x+1)^2}\right]\left(\frac{1}{y+1}+\frac{1}{x+1}\right) \\ &= \frac{1}{y+x+xy}-\frac{6x}{(y+x+xy)^2}+\frac{4x^2}{(y+x+xy)^3} \\ &\quad +\frac{x}{y+x+xy}+\frac{5x^2+14x+11}{(x+1)^3}\,\frac{x^2}{(y+x+xy)^2}-\frac{4x^2+11x+8}{(x+1)^3}\,\frac{x^3}{(y+x+xy)^3} \\ &\quad -\left[1+\frac{1}{(x+1)^2}\right]\left(\frac{1}{y+1}+\frac{1}{x+1}\right) \\ &= \frac{1}{y+x}-\frac{6x}{(y+x)^2}+\frac{4x^2}{(y+x)^3}-h(x,y)\,, \end{aligned}$$

where

$$\begin{aligned} h(x,y) &= \frac{1}{y+x}-\frac{6x}{(y+x)^2}+\frac{4x^2}{(y+x)^3}-\frac{1}{y+x+xy}+\frac{6x}{(y+x+xy)^2}-\frac{4x^2}{(y+x+xy)^3} \\ &\quad -\frac{x}{y+x+xy}-\frac{5x^2+14x+11}{(x+1)^3}\,\frac{x^2}{(y+x+xy)^2}+\frac{4x^2+11x+8}{(x+1)^3}\,\frac{x^3}{(y+x+xy)^3} \\ &\quad +\left[1+\frac{1}{(x+1)^2}\right]\left(\frac{1}{y+1}+\frac{1}{x+1}\right) \\ &= \frac{1}{y+x}-\frac{1}{y+x+xy}-6x\left[\frac{1}{(y+x)^2}-\frac{1}{(y+x+xy)^2}\right]+4x^2\left[\frac{1}{(y+x)^3}-\frac{1}{(y+x+xy)^3}\right] \\ &\quad +\left\{\left[\frac{((4x+11)x+8)x}{y+x+xy}-(5x+14)x-11\right]\frac{x}{(y+x+xy)(x+1)^3}-1\right\}\frac{x}{y+x+xy} \\ &\quad +\left[1+\frac{1}{(x+1)^2}\right]\left(\frac{1}{y+1}+\frac{1}{x+1}\right) \end{aligned}$$

is bounded and continuously differentiable for $(x,y)\in(0,\infty)\times[0,\infty)$.

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