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Interpolating given tangent vectors or curvatures by preprocessed incenter subdivision scheme

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Abstract

This paper introduces a preprocessed incenter subdivision scheme to make the limit curves interpolate G^1 or G^2 Hermite data. The key idea is to add a preprocessed step, changing the compute rule of new inserted points and their corresponding tangent vectors in the first subdivision step of the incenter subdivision scheme[4]. Moreover, we prove that the limit curves generated by our algorithm interpolate given tangent vectors or curvatures at given points, with some mathematical analyses techniques. Numerical examples elucidate the validity of our approach.

Keywords: incenter subdivision scheme, preprocess, tangent vectors, curvature.

1 Introduction

Subdivision schemes are efficient and adaptive tools for generating smooth curves and surfaces from discrete data using recursive rules. The algorithms are popularly used in graphical modeling, animation industry and CAD/CAM([6],[9],[11]) because of their simplicity, stability and high flexibility in computer applications. For the linear subdivision methods, such as Chaikin's corner cutting algorithm[3], four point subdivision scheme[5] and non-uniform four point subdivision scheme[7], the refinement rules do not change during the subdivision process. Therefore, such schemes are simple to be implemented and easy to analyze their convergence and smoothness, but it is hard to control the shape of the limit curve.

To produce certain classes of shapes, such as a perfect circle, the nonlinear subdivision schemes are developed([1],[2],[4],[8],[10]). For example, Albrecht and Romani[1] are concerned with the convexity preserving interpolatory subdivision with conic precision. Aspert et al.[2] presents a nonlinear subdivision algorithm keeping the complexity acceptable. Romani[8] offers a tension-controlled 2-point Hermite interpolatory subdivision scheme to reproduce circles, originated from a sequence of sample points with any arbitrary spacing and suitably selected first and second derivatives. Yang[10] provides a normal based subdivision scheme for curve design, in which the limit curve is G^1 smooth with wide ranges of free parameters and normal vectors at selected vertexes are efficiently interpolated to keep the shape preserved. Deng and Wang [4] put forward an incenter subdivision scheme based on bi-arc interpolation. The scheme makes its limit curve interpolate given initial points, but in general does not interpolate tangent vectors or curvatures at given points.

Enlightened by [4], this paper introduces an algorithm to interpolate G^1 or G^2 Hermite data by our preprocessed incenter subdivision scheme. In order to interpolate given tangents or curvatures, the rules to compute new points and tangent vectors adjacent to the given points in the first subdivision step are revised. After preprocessing, the incenter subdivision scheme is applied to generate the desired limit curves from Hermite data.

The remaining part of the paper is arranged as follows. Section 2 shows incenter subdivision schemes in detail and provides the main results for our preprocessed incenter subdivision scheme. Section 3 introduces the detailed preprocessing procedure, preparing for the proof of main results in Section 4. Numerical examples in Section 5 elucidate the efficiency of our approach. Finally, we make conclusions and propose the future work in Section 6.

2 Preliminaries and Main Results

2.1 Preliminaries for Incenter Subdivision Scheme

Given a sequence of initial control points $\{\boldsymbol{p}_i^0\}$ and tangents $\{\boldsymbol{T}_i^0\}$, the new inserted points $\{\boldsymbol{p}_i^k\}$ and tangents $\{\boldsymbol{T}_i^k\}(k \geq 1)$ deduced by the incenter subdivision scheme [4] are defined in the following two steps.

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Step 1: Compute new inserted points.

$$\begin{cases} \boldsymbol{p}_{2i}^{k+1} = \boldsymbol{p}_i^k, \\ \boldsymbol{p}_{2i+1}^{k+1} = \boldsymbol{p}_i^k + \dfrac{\sin(0.5\beta_i^k)}{\sin(0.5\alpha_i^k + 0.5\beta_i^k)} R(-0.5\alpha_i^k)(\boldsymbol{p}_{i+1}^k - \boldsymbol{p}_i^k), \end{cases} \tag{1}$$

where α_i^k is the angle from $\boldsymbol{T}_i^k$ to $\boldsymbol{p}_i^k\boldsymbol{p}_{i+1}^k$, β_i^k is the angle from $\boldsymbol{p}_i^k\boldsymbol{p}_{i+1}^k$ to $\boldsymbol{T}_{i+1}^k$. And for a vector $\boldsymbol{V}$, $R(\theta)\boldsymbol{V}$ rotates the vector $\boldsymbol{V}$ counter-clockwise through an angle θ.

Step 2: Compute new tangents.

Firstly, define a set of provisional tangents:

$$\begin{cases} \boldsymbol{U}_{2i}^{k+1} = \boldsymbol{T}_i^k, \\ \boldsymbol{U}_{2i+1}^{k+1} = \dfrac{\boldsymbol{p}_{i+1}^k - \boldsymbol{p}_i^k}{\|\boldsymbol{p}_{i+1}^k - \boldsymbol{p}_i^k\|}. \end{cases} \tag{2}$$

Secondly, calculate new tangents in each point by rotating provisional tangents through an angle $-\delta_i^k$:

$$\boldsymbol{T}_i^k = R(-\delta_i^k)\boldsymbol{U}_i^k. \tag{3}$$

For δ_i^k, please refer to the Definition 2.3 of reference [4].

According to the above-mentioned incenter subdivision scheme, the generated limit curves interpolate given initial points, but in general do not interpolate tangent vectors or curvatures at given points.

2.2 Main Results

To make the limit curves interpolate G^1 or G^2 Hermite data, a preprocessing procedure is applied to compute the new inserted points $\boldsymbol{p}_{2i-1}^1$ and $\boldsymbol{p}_{2i+1}^1$, and to redefine the corresponding tangent vectors $\boldsymbol{T}_{2i-1}^1$ and $\boldsymbol{T}_{2i+1}^1$ tangent to curvature circle of point $\boldsymbol{p}_i^0$ and $\boldsymbol{p}_{i+1}^0$ respectively. After this preprocessing, we continue to apply the aforementioned incenter subdivision rules to create the limited curves. Then, the limited curves have the following two results.

Theorem 1. The limit curve, generated by our preprocessed incenter subdivision scheme, interpolates given tangent vector.

Theorem 2. The limit curve, generated by our preprocessed incenter subdivision scheme, interpolates given tangent vector and curvature at given point.

In order to prove Theorem 1 and Theorem 2, the preprocessing procedure will be introduced in detail in the following section.

3 Preprocessing Procedure before Applying Incenter Subdivision Scheme

This section will divide the process into two parts. Section 3.1 presents the determination of the curvature circle of point $\boldsymbol{p}_i^0$, and Section 3.2 provides the computation rule of new inserted points $\boldsymbol{p}_{2i-1}^1$ and $\boldsymbol{p}_{2i+1}^1$ and their corresponding tangent vectors $\boldsymbol{T}_{2i-1}^1$ and $\boldsymbol{T}_{2i+1}^1$.

3.1 Determining Curvature Circle at Given Points

Firstly, determine circle $\odot\boldsymbol{o}_1$ passing through $\boldsymbol{p}_i^0$ and $\boldsymbol{p}_{i-1}^0$ and tangent to $\boldsymbol{T}_i^0$, and compute its radius r_1 (see the Fig.1). So we have

$$r_1 = \frac{\|\boldsymbol{p}_{i-1}^0\boldsymbol{p}_i^0\|}{2\sin|\beta_{i-1}^0|}.$$

Similarly, determine circle $\odot\boldsymbol{o}_2$ passing through $\boldsymbol{p}_i^0$ and $\boldsymbol{p}_{i+1}^0$ and tangent to $\boldsymbol{T}_i^0$ in the Fig.1, and its radius r_2 is

$$r_2 = \frac{\|\boldsymbol{p}_i^0\boldsymbol{p}_{i+1}^0\|}{2\sin|\alpha_i^0|}.$$

Then, we obtain the curvature circle $\odot\boldsymbol{o}$ at given point $\boldsymbol{p}_i^0$ tangent to given $\boldsymbol{T}_i^0$, and define its radius r to be

$$r = \sqrt{r_1 r_2}. \tag{4}$$

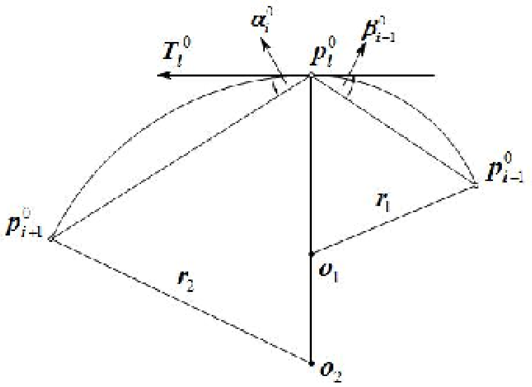

Figure 1: The definition of curvature circle at given point $\boldsymbol{p}_i^0$

3.2 Calculating new inserted points adjacent to given point and corresponding tangent vecotrs

In this section, the computation rule of new inserted points $\boldsymbol{p}_{2i-1}^1$ and $\boldsymbol{p}_{2i+1}^1$ and their corresponding tangent vectors $\boldsymbol{T}_{2i-1}^1$ and $\boldsymbol{T}_{2i+1}^1$ is provided. In connection with computing the new inserted points $\boldsymbol{p}_{2i-1}^1$, first, we define it in the following two conditions.

Condition 1: if the angle bisector of $\angle \boldsymbol{T}_{i-1}^0\boldsymbol{p}_{i-1}^0\boldsymbol{p}_i^0$ and curvature circle $\odot\boldsymbol{o}$ at given point $\boldsymbol{p}_i^0$ have the crosspoint, then the crosspoint is the new inserted point $\boldsymbol{p}_{2i-1}^1$ (see Fig.2(a));

Condition 2: if there doesn't exist the crosspoint between the angle bisector of $\angle \boldsymbol{T}_{i-1}^0\boldsymbol{p}_{i-1}^0\boldsymbol{p}_i^0$ and curvature circle $\odot\boldsymbol{o}$, construct the tangent line $\boldsymbol{p}_{i-1}^0 C$ of the circle $\odot\boldsymbol{o}$ at given point $\boldsymbol{p}_i^0$. Then, the new inserted point $\boldsymbol{p}_{2i-1}^1$ is the intersection point between the angle bisector of $\angle C\boldsymbol{p}_{i-1}^0\boldsymbol{p}_i^0$ and $\odot\boldsymbol{o}$ (see Fig.2(b)).

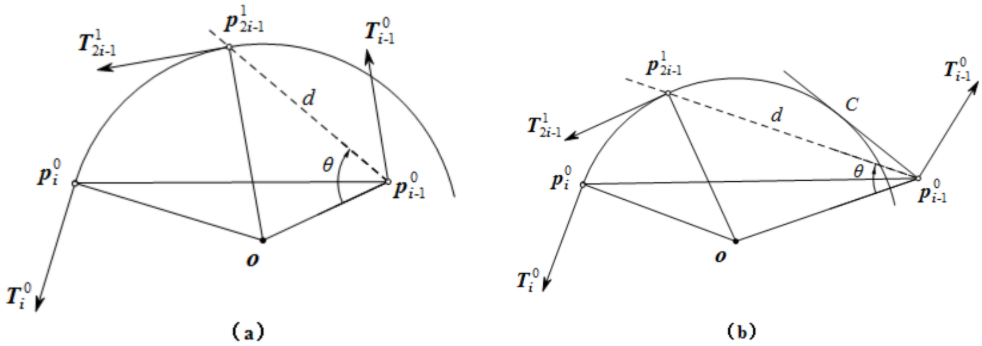

Figure 2: The calculation rule of new inserted point $\boldsymbol{p}_{2i-1}^1$:(a)$\boldsymbol{p}_{2i-1}^1$ is the intersection point between the angle bisector of $\angle \boldsymbol{T}_{i-1}^0\boldsymbol{p}_{i-1}^0\boldsymbol{p}_i^0$ and $\odot\boldsymbol{o}$; (b)$\boldsymbol{p}_{2i-1}^1$ is the angle bisector of $\angle C\boldsymbol{p}_{i-1}^0\boldsymbol{p}_i^0$ and $\odot\boldsymbol{o}$.

Next, compute the angle θ from the edge $\boldsymbol{p}_{i-1}^0\boldsymbol{o}$ to the edge $\boldsymbol{p}_{i-1}^0\boldsymbol{p}_{2i-1}^1$. As can be seen from Fig.2, θ satisfies the equality

$$\theta = \angle \boldsymbol{o}\boldsymbol{p}_{i-1}^0\boldsymbol{p}_i^0 + \angle \boldsymbol{p}_i^0\boldsymbol{p}_{i-1}^0\boldsymbol{p}_{2i-1}^1.$$

Here, $\angle \boldsymbol{o}\boldsymbol{p}_{i-1}^0\boldsymbol{p}_i^0$ is the angle from $\boldsymbol{p}_{i-1}^0\boldsymbol{o}$ to $\boldsymbol{p}_{i-1}^0\boldsymbol{p}_i^0$, and $\angle \boldsymbol{p}_i^0\boldsymbol{p}_{i-1}^0\boldsymbol{p}_{2i-1}^1$ is the angle from $\boldsymbol{p}_{i-1}^0\boldsymbol{p}_i^0$ to $\boldsymbol{p}_{i-1}^0\boldsymbol{p}_{2i-1}^1$. That is, $\angle \boldsymbol{p}_i^0\boldsymbol{p}_{i-1}^0\boldsymbol{p}_{2i-1}^1$ is acquired in the Fig.2(a) by

$$\angle \boldsymbol{p}_i^0\boldsymbol{p}_{i-1}^0\boldsymbol{p}_{2i-1}^1 = -\frac{1}{2}\alpha_{i-1}^0,$$

or obtained in the Fig.2(b) by

$$\angle \boldsymbol{p}_i^0\boldsymbol{p}_{i-1}^0\boldsymbol{p}_{2i-1}^1 = \frac{1}{2}(\angle \boldsymbol{o}\boldsymbol{p}_{i-1}^0 C - \angle \boldsymbol{o}\boldsymbol{p}_{i-1}^0\boldsymbol{p}_i^0),$$

and $\angle \boldsymbol{o}\boldsymbol{p}_{i-1}^0 C$ is the angle from the edge $\boldsymbol{o}\boldsymbol{p}_{i-1}^0$ to $\boldsymbol{p}_{i-1}^0 C$ calculated by $sin|\angle \boldsymbol{o}\boldsymbol{p}_{i-1}^0 C| = \frac{r}{||\boldsymbol{o}\boldsymbol{p}_{i-1}^0||}$.

Then, denote d as the length of edge $\boldsymbol{p}_{i-1}^0\boldsymbol{p}_{2i-1}^1$, and d can be obtained by

$$d = ||\boldsymbol{p}_{i-1}^0\boldsymbol{p}_{2i-1}^1|| = ||\boldsymbol{o}\boldsymbol{p}_{i-1}^0||cos\theta + \sqrt{r^2 - (||\boldsymbol{o}\boldsymbol{p}_{i-1}^0||sin\theta)^2}.$$

Finally, the new inserted point $\boldsymbol{p}^1_{2i-1}$ is

$$\boldsymbol{p}^1_{2i-1} = \boldsymbol{p}^0_{i-1} + dR(\theta)\frac{\boldsymbol{o}-\boldsymbol{p}^0_{i-1}}{||\boldsymbol{o}-\boldsymbol{p}^0_{i-1}||}. \tag{5}$$

Accordingly, the new tangent vector $\boldsymbol{T}^1_{2i-1}$ at $\boldsymbol{p}^1_{2i-1}$ is defined by

$$\boldsymbol{T}^1_{2i-1} = R(-\frac{\pi}{2})\frac{\boldsymbol{o}-\boldsymbol{p}^1_{2i-1}}{||\boldsymbol{o}-\boldsymbol{p}^1_{2i-1}||}. \tag{6}$$

Similarly, let $\overline{\boldsymbol{T}}^0_{i+1} = -\boldsymbol{T}^0_{i+1}$, and we define the new inserted point $\boldsymbol{p}^1_{2i+1}$ to be the crosspoint between the angle bisector of $\angle\overline{\boldsymbol{T}}^0_{i+1}\boldsymbol{p}^0_{i+1}\boldsymbol{p}^0_i$ and $\odot\boldsymbol{o}$ (see Fig.3(a)). If there doesn't exist the intersection point between the angle bisector of $\angle\overline{\boldsymbol{T}}^0_{i+1}\boldsymbol{p}^0_{i+1}\boldsymbol{p}^0_i$ and $\odot\boldsymbol{o}$, construct the tangent $\boldsymbol{p}^0_{i+1}D$ to the circle $\odot\boldsymbol{o}$ at the point $\boldsymbol{p}^0_i$. Then, $\boldsymbol{p}^1_{2i+1}$ is the intersection point between the angle bisector of $\angle D\boldsymbol{p}^0_{i+1}\boldsymbol{p}^0_i$ and $\odot\boldsymbol{o}$ (see Fig.3(b)). At last, with same compute rule of $\boldsymbol{p}^1_{2i-1}$ and $\boldsymbol{T}^1_{2i-1}$, we acquire the new inserted point $\boldsymbol{p}^1_{2i+1}$ and its new tangent vector $\boldsymbol{T}^1_{2i+1}$.

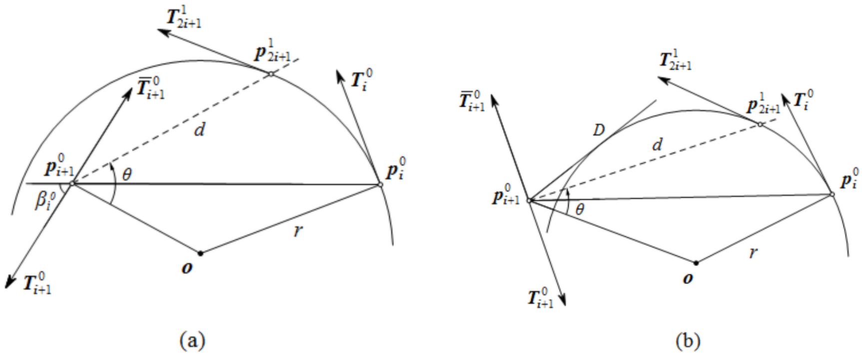

Figure 3: The calculation rule of new inserted point $\boldsymbol{p}^1_{2i+1}$:(a)$\boldsymbol{p}^1_{2i+1}$ is the intersection point between the angle bisector of $\angle\overline{\boldsymbol{T}}^0_{i+1}\boldsymbol{p}^0_{i+1}\boldsymbol{p}^0_i$ and $\odot\boldsymbol{o}$; (b)$\boldsymbol{p}^1_{2i+1}$ is the angle bisector of $\angle D\boldsymbol{p}^0_{i+1}\boldsymbol{p}^0_i$ and $\odot\boldsymbol{o}$.

Remark 3. Only the compute rule of new inserted points $\boldsymbol{p}^1_{2i-1}$ and $\boldsymbol{p}^1_{2i+1}$ is revised, and the other new points and their tangents in each step are still obtained by the incenter subdivision scheme[4].

4 Proof of Main Results

4.1 Proof of Theorem 1

Assume that $\boldsymbol{T}^0_i$ is the given tangent vector at initial point $\boldsymbol{p}^0_i$, and let $\boldsymbol{p}^k_{2^k i\pm1}(k \geq 1)$ be the new inserted points. Because the conclusion for $\boldsymbol{p}^k_{2^k i+1}$ and $\boldsymbol{T}^k_{2^k i+1}$ can be proved in a similar way to that for $\boldsymbol{p}^k_{2^k i-1}$ and $\boldsymbol{T}^k_{2^k i-1}$, only the result of $\boldsymbol{p}^k_{2^k i-1}$ and $\boldsymbol{T}^k_{2^k i-1}$ will be inferred. Next we will deduce that $\boldsymbol{p}^k_{2^k i-1}$ is on the curvature circle and its tangent vector $\boldsymbol{T}^k_{2^k i-1}$ is tangent to the circle by mathematical induction.

Obviously, the statement holds for $k = 1$ on account of our preprocessed incenter subdivision scheme. For $k = n$, suppose the new inserted points $\boldsymbol{p}^n_{2^n i-1}$ are on the curvature circle and their tangent vectors $\boldsymbol{T}^n_{2^n i-1}$ are tangent to the circle. Then we deduce that the conclusion is suitable for $k = n+1$ as follows.

As can be seen from Fig.4, we know that $\boldsymbol{p}^{n+1}_{2^{n+1}i-1}$ is the crosspoint between the angle bisectors of $\angle\overline{\boldsymbol{T}}^n_{2^n i}\boldsymbol{p}^n_{2^n i}\boldsymbol{p}^n_{2^n i-1}$ and $\angle\boldsymbol{p}^n_{2^n i}\boldsymbol{p}^n_{2^n i-1}\boldsymbol{T}^n_{2^n i-1}$. Owing to the fact that $\boldsymbol{T}^n_{2^n i}$ and $\boldsymbol{T}^n_{2^n i-1}$ are both tangent to the circle, we have

$$\boldsymbol{op}^{n+1}_{2^{n+1}i-1} \perp \boldsymbol{p}^n_{2^n i}\boldsymbol{p}^n_{2^n i-1},\ \angle\boldsymbol{p}^n_{2^n i}\boldsymbol{op}^{n+1}_{2^{n+1}i-1} = |\beta^n_{2^n i-1}|.$$

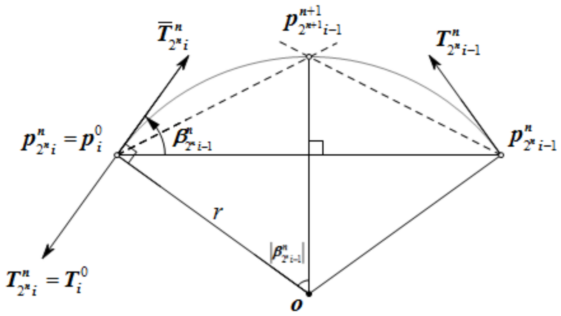

Figure 4: Analysis of Theorem 1—the limit curve interpolates given tangent vector

So in $\triangle p^n_{2^n i} o p^{n+1}_{2^{n+1}i-1}$, we find that

$$\begin{aligned}|op^{n+1}_{2^{n+1}i-1}| &= r\cos|\beta^n_{2^n i-1}| + r\sin|\beta^n_{2^n i-1}|\tan(0.5|\beta^n_{2^n i-1}|)\\ &= r\cos|\beta^n_{2^n i-1}| + 2r\sin^2(0.5|\beta^n_{2^n i-1}|)\\ &= r\cos|\beta^n_{2^n i-1}| + r(1-\cos|\beta^n_{2^n i-1}|)\\ &= r,\end{aligned} \tag{7}$$

which implies the new inserted point $p^{n+1}_{2^{n+1}i-1}$ is on the curvature circle.

Besides, due to the equality $r^n_{2^n i-1} = r^n_{2^n i} = r$, it can be found that the rotation angle $\delta^{n+1}_{2^{n+1}i-1} = 0$. As a result, $T^{n+1}_{2^{n+1}i-1}$ is tangent to $\odot o$ at point $p^{n+1}_{2^{n+1}i-1}$ based on the equality $T^{n+1}_{2^{n+1}i-1} = U^{n+1}_{2^{n+1}i-1}$.

From the aforesaid analyses, we know that the new inserted points $p^k_{2^k i\pm1}$ are always on the curvature circle, and their tangent vectors $T^k_{2^k i\pm1}$ are tangent to $\odot o$. That is, $\lim_{k\to\infty} T^k_{2^k i\pm1} = T^0_i$ indicating that Theorem 1 is correct.

Remark 4. With regard to the other new inserted points, they are also on the same circular arc segments. For a detailed proof, please see the reference[4].

4.2 Proof of Theorem 2

Given tangent vector T^0_i and curvature ρ at the initial point p^0_i, the radius of curvature can be obtained by $r = \frac{1}{\rho}$. Since new inserted points $p^k_{2^k i\pm1}$ are always on the circle, we have $r^k_{2^k i\pm1} = r$, namely,

$$\lim_{k\to\infty} r^k_{2^k i\pm1} = r.$$

In addition, by Theorem 1 we find that

$$\lim_{k\to\infty} T^k_{2^k i\pm1} = T^0_i.$$

Consequently, the limit curve generated by our preprocessed incenter subdivision scheme interpolates given tangent vector T^0_i and curvature ρ at given point p^0_i, so proving the statement of Theorem 2.

5 Numerical Examples

In this section we present several examples to show the effect of our preprocessed incenter subdivision scheme. Of all the graphs, red curves, green curves, black segments and purple segments represent generated subdivision curves, curvatures, given tangent vectors and given curvatures respectively.

Fig.5 shows that the limit curve is generated by an open curve interpolating given tangent vector $(0.2822, 0.9594)$, and the initial points are $(3,6)$, $(6,8)$, $(9,8)$, $(8,5)$, $(10,2)$, $(8,0)$, $(4,2)$.

Fig.6 shows that the limit curves generated by closed curves interpolate given tangent vectors, and the initial points are $(750,120)$, $(850,300)$, $(700,500)$, $(500,500)$, $(350,300)$, $(450,120)$, $(600,240)$. On top of that, the first two graphics are given tangents in different points and the last two graphics are given different tangents in the same point.

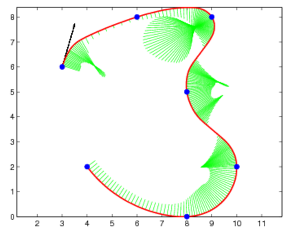

Figure 5: Interpolating given tangent vector

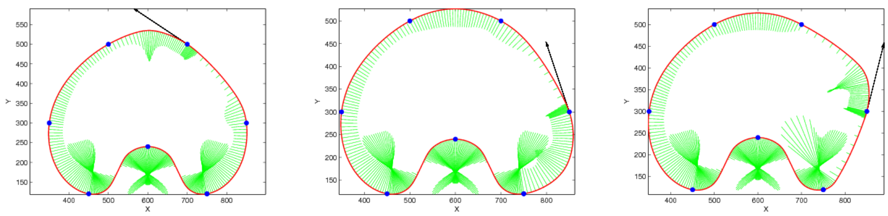

Figure 6: Interpolating given tangent vectors: the first two graphics are given tangents in different points; the last two graphics are given different tangents in the same point.

In Fig.7, two limit curves are produced by the initial points $(-1,4)$, $(1,1)$, $(7,5)$, $(4,11)$, interpolating the same tangent vector $(0.2425, 0.9701)$ and different curvatures $\rho_1 = 0.25$ and $\rho_2 = 0.15$ at point $(7,5)$ respectively.

In Fig.8, the initial points are $(0,400)$, $(500,-100)$, $(1000,400)$, $(750,750)$, $(500,500)$, $(250,750)$. Two generated limit curves interpolate different tangent vectors and same curvature at given point.

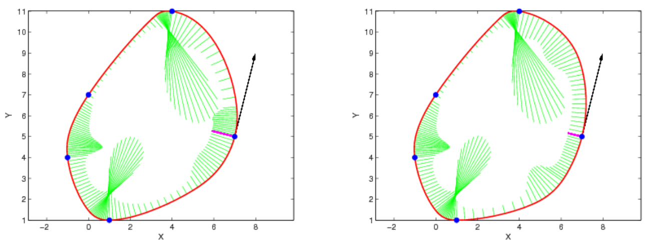

Figure 7: Interpolating same given tangent vector and different given curvatures

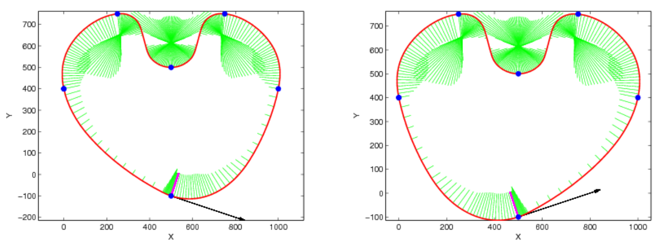

Figure 8: Interpolating different given tangent vectors and same given curvature

6 Conclusions and future work

In this paper, in order to achieve the limit curves interpolating given tangent vectors or curvatures, we revise the compute rule of new inserted points and redefine their tangent vectors in the first subdivision step of incenter subdivision scheme. Both theoretical analyses and examples validate that our scheme indeed interpolates G^1 or G^2 Hermite data. However, only one point is being interpolated by G^1 or G^2 Hermite data. So, how to find effective methods to interpolate given data at a few points is our future work.

Acknowledgement

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