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Polynomial approximation on pyramids, cones and solids of rotation

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Abstract

Given a planar compact set Ω where a weakly admissible mesh (WAM) is known, we compute WAMs and the corresponding discrete extremal sets for polynomial interpolation on solid (even truncated) cones with base Ω (with pyramids as a special case), and on solids corresponding to the rotation of Ω around an external coplanar axis by a given angle.

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1 Introduction

In the seminal paper [8], the notion of admissible mesh for multivariate polynomial approximation has been introduced. A (weakly) admissible mesh is a finite norming set, with respect to the infinity-norm, for polynomials of a fixed degree on a given (polynomial determining) compact set. In recent years, there has been increasing attention to this notion in the approximation theory and numerical literature, due to the deep connections with multivariate polynomial interpolation and approximation.

For example, in [8] it has been shown that WAMs are nearly optimal for polynomial least squares approximation in the uniform norm. On the other hand, discrete extremal sets (of Fekete and Leja type) extracted from such meshes show good Lebesgue constants and behave asymptotically as the corresponding continuous extremal sets; we refer the reader, e.g., to the technical papers [2, 3, 4, 18], and to the excellent survey [1] on the state of the art in multivariate polynomial interpolation and approximation. It is also worth observing that such discrete extremal sets have begun to play a role in the numerical PDEs context, concerning spectral element and collocation methods; cf., e.g., [14, 21].

We recall that a weakly admissible mesh (WAM) is a sequence of finite subsets of a multidimensional compact set, say $\mathcal{A}_n \subset K \subset \mathbb{C}^d$, such that

$$\|p\|_K \le C(\mathcal{A}_n)\,\|p\|_{\mathcal{A}_n}\,, \ \ \forall p \in \mathbb{P}_n^d(K)\,, \tag{1}$$

where both $C(\mathcal{A}_n)$ and $\mathrm{card}(\mathcal{A}_n)$ increase at most polynomially with n; here and below, $\mathbb{P}_n^d(K)$ denotes the space of d-variate polynomials of degree not exceeding n (restricted to K), and $\|f\|_X$ the sup-norm of a function f bounded on the (discrete or continuous) set X. Observe that necessarily $\mathrm{card}(\mathcal{A}_n) \ge \dim(\mathbb{P}_n^d(K))$.

When $C(\mathcal{A}_n)$ is bounded we speak of an admissible mesh. Among their properties, we quote that WAMs are preserved by affine transformations, and can be constructed incrementally by finite union and product. Moreover, we recall that unisolvent interpolation point sets, with slowly (at most polynomially) increasing Lebesgue constant, are WAMs, with $C(\mathcal{A}_n)$ equal to the Lebesgue constant and $\mathrm{card}(\mathcal{A}_n) = \dim(\mathbb{P}_n^d(K))$. Concerning these and other basic features of WAMs, we refer the reader to [4, 8].

In the present note, which is mainly of computational character, we construct real 3-dimensional WAMs, on two classes of solid compact sets corresponding to a given planar compact, say Ω, where a WAM is known.

The first class are solid cones with base Ω and vertex x^*, i.e., the sets formed by all the segments connecting x^* with a point of Ω (pyramids, i.e., cones with polygonal base, being a subclass), along with the truncated cones obtained by cutting with a plane parallel to the base. The second are solids of rotation with cross section Ω and (external) axis α, i.e., the sets obtained by the rotation of Ω by a given angle, even smaller than 2π, around a coplanar line α. In this case, we resort to some recent results on trigonometric interpolation on subintervals of the period, cf. [6, 9].

We show that the (numerically evaluated) infinity-norms of the polynomial least squares approximation corresponding to the whole meshes, are much lower than the theoretical estimates given in [8]. Moreover, using the results developed in [2, 3, 4, 18] and the numerical code [19], we are able to compute from the meshes discrete extremal sets for polynomial interpolation, along with numerical estimates of their Lebesgue constants.

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2 Cones and solids of rotation

In this section we state and prove the main result, in a geometric fashion. We do not make here a distinction between cones and pyramids, since the latter are cones with a polygonal base.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a planar compact set where a 2-dimensional WAM, say $\mathcal{A}_n$, is known, cf. (1). Let x^* be a point in $\mathbb{R}^3$ not belonging to the plane of Ω; let α be a line in $\mathbb{R}^3$ lying on the plane of Ω and not intersecting Ω (or intersecting Ω only at the boundary), and let $\theta^* \in (0, 2\pi]$ a given angle. Then, the following hold:*

(i) (cones) *the cone $\mathcal{C}$ with base Ω and vertex x^*, that is the pencil of the segments joining x^* with each point of Ω, has a WAM, say $\mathcal{B}_n$, with $C(\mathcal{B}_n) = \mathcal{O}(C(\mathcal{A}_n)\log n)$ and $card(\mathcal{B}_n) = 1 + ncard(\mathcal{A}_n)$, given by the union of the $n+1$ Chebyshev-Lobatto points of the segments joining x^* with each of the points of $\mathcal{A}_n$; if we consider the truncated cone obtained by cutting the cone with a plane parallel to the base, then the WAM is given by the union of the Chebyshev-Lobatto points of the cut segments, and has cardinality $(n+1)card(\mathcal{A}_n)$;*

(ii) (solids of rotation) *the solid of rotation $\mathcal{R}$ obtained by rotating Ω around the axis α by an angle θ^* has a WAM, say $\mathcal{B}_n$, with $C(\mathcal{B}_n) = \mathcal{O}(C(\mathcal{A}_n)\log n)$ and $card(\mathcal{B}_n) = (2n+1)card(\mathcal{A}_n)$, given by the union of the $2n+1$ copies of $\mathcal{A}_n$ corresponding to rotating $\mathcal{A}_n$ by the angles*

$$\theta_j = \frac{\theta^*}{2} + 2\arcsin\left(\xi_j \sin(\theta^*/4)\right) \in (0, \theta^*)\,, \tag{2}$$

where $\{\xi_j\}$ are the classical Chebyshev points in $(-1,1)$, i.e., the zeros of the Chebyshev polynomial $T_{2n+1}(\cdot)$

$$\xi_j = \cos\left(\frac{(2j-1)\pi}{2(2n+1)}\right)\,,\ \ j = 1, 2, \ldots, 2n+1\,.$$

Proof. We first prove (i). Given a polynomial $p \in \mathbb{P}_n^3$ and a point x belonging to the cone, the restriction of p to the pencil segment $\sigma(x)$ corresponding to x, is a univariate polynomial and thus the following inequality holds

$$|p(x)| \le \Lambda_n \|p\|_{X_n(\sigma(x))}\,,\ \ X_n(\sigma(x)) = \{\tau_j(\sigma(x)),\, j = 0, \ldots, n\}\,, \tag{3}$$

where $\{\tau_j(\sigma(x))\}$ are $n+1$ Chebyshev-Lobatto points of the segment (ordered in such a way, for example, $\tau_0 = x^*$ and $\tau_n \in \Omega$), and $\Lambda_n = \mathcal{O}(\log n)$ the Lebesgue constant of Chebyshev-Lobatto interpolation of degree n (which is invariant under affine transformations).

Then, we get

$$\|p\|_{\mathcal{C}} \le \Lambda_n \|p\|_{\bigcup_x X_n(\sigma(x))} = \Lambda_n \|p\|_{\bigcup_j \Omega_j}\,, \tag{4}$$

where

$$\Omega_j = \{\tau_j(\sigma(x)),\, x \in \mathcal{C}\} = \{\tau_j(\sigma(y)),\, y \in \Omega\}\,,\ \ j = 0, \ldots, n. \tag{5}$$

Observe that, by the intercept theorem of Thales of Miletus applied to each pair of segments of the pencil, the points $\tau_j(\sigma(x))$ for a given j are all coplanar, thus the sets Ω_j belong to planes parallel to the plane of Ω, and are affine transformations of Ω. It follows that a polynomial $p \in \mathbb{P}_n^3$ restricted to Ω_j is a bivariate polynomial satisfying the polynomial inequality

$$\|p\|_{\Omega_j} \le C(\mathcal{A}_n)\|p\|_{Y_{n,j}}\,,\ \ Y_{n,j} = \{\tau_j(\sigma(y)),\, y \in \mathcal{A}_n\}\,,\ \ j = 0, \ldots, n\,,$$

since the constant in (1) is invariant under affine transformations, and finally by (4)

$$\|p\|_{\mathcal{C}} \le \Lambda_n \|p\|_{\bigcup_j \Omega_j} \le \Lambda_n\, C(\mathcal{A}_n)\|p\|_{\bigcup_j Y_{n,j}}\,, \tag{6}$$

i.e., $\mathcal{B}_n = \bigcup_{j=0}^{n} Y_{n,j} = \bigcup_{y \in \mathcal{A}_n} X_n(\sigma(y))$ is a WAM for the cone $\mathcal{C}$. The assertion on the cardinality of $\mathcal{B}_n$ follows immediately by subtracting the repetitions of the the vertex. In the case of a truncated cone, we can follow exactly the reasoning above, with the only difference that the Chebyshev-Lobatto points are those of the subsegments corresponding to the cut.

To prove (ii), we consider cylindrical coordinates, say $((r,t),\theta)$,

$$(x_1, x_2, x_3) = \phi((r,t),\theta) = (r\cos\theta, r\sin\theta, t)\,,\ \ \phi : \Omega \times [0,\theta^*] \to \mathcal{R}\,,$$

with respect to the (oriented) axis α that, with no loss of generality can be taken as the x_3 axis, $x = (x_1, x_2, x_3)$. In such coordinates, a polynomial $p \in \mathbb{P}_n^3$ becomes a tensor product polynomial, algebraic in (r,t), and trigonometric in θ, say $q((r,t),\theta)$, with $q \in \mathbb{P}_n^2 \bigotimes \mathbb{T}_n$, where $\mathbb{T}_n$ denotes the space of univariate trigonometric polynomials of degree not exceeding n.

Since the underline transformation into cylindrical coordinates is surjective, $\mathcal{R} = \phi(\Omega \times [0,\theta^*])$, we have

$$\|p\|_{\mathcal{R}} = \|q\|_{\Omega \times [0,\theta^*]}\,. \tag{7}$$

Now, for every $((r,t),\theta) \in \Omega \times [0,\theta^*]$ we can write the chain of inequalities

$$|q((r,t),\theta)| \le C(\mathcal{A}_n) \max_{(r,t)\in\mathcal{A}_n} |q((r,t),\theta)|$$

$$\le C(\mathcal{A}_n) \max_{(r,t)\in\mathcal{A}_n} \Lambda_n \max_{\theta\in\Theta_n} |q((r,t),\theta)| = C(\mathcal{A}_n)\,\Lambda_n \|q\|_{\mathcal{A}_n \times \Theta_n}\,, \tag{8}$$

where $\Theta_n = \{\theta_j\,,\, j = 1, \ldots, 2n+1\}$ (cf. (2)) is a set of suitable angular nodes for trigonometric interpolation on subintervals of the period, and $\Lambda_n = \mathcal{O}(\log n)$ their Lebesgue constant, which is invariant with respect to the angular interval; cf. [6, 9]. From (7)-(8) immediately follows

$$\|p\|_{\mathcal{R}} \le C(\mathcal{A}_n)\,\Lambda_n\,\|p\|_{\phi(\mathcal{A}_n\times\Theta_n)}\,, \tag{9}$$

i.e., $\mathcal{B}_n = \phi(\mathcal{A}_n \times \Theta_n)$ (that corresponds to $2n+1$ copies of $\mathcal{A}_n$ rotated by the angles θ_j) is a WAM for the solid of rotation $\mathcal{R}$. □

Remark 1. It is clear that (i) of Theorem 1 can be extended in a straightforward way to "cylindroids", that are truncated cones where the vertex x^* is taken at infinity. In such instances, the segments $\sigma(x)$ are all parallel and the sets Ω_j, $Y_{n,j}$ are simply translations of Ω and $\mathcal{A}_n$, respectively (cf. [10] for the case of the standard cylinder).

Remark 2. Since Theorem 1 refers to planar compacts where a WAM is known, it is worth recalling that any polynomial determining compact $\Omega \subset \mathbb{C}^2$ admits trivially a WAM with $(n+1)(n+2)/2$ cardinality (its Fekete points), but also an admissible mesh of cardinality $\mathcal{O}((n\log n)^2)$, an existence result recently proved in [1, Prop.23]. Such admissible meshes, however, being formed by Fekete points of suitable degree, are difficult to compute. On the other hand, real (weakly) admissible meshes with optimal cardinality $\mathcal{O}(n^2)$ are known constructively in several instances, not only on basic geometries, such as the triangle and the disk (cf. [5, 6]), but for example also on convex and concave polygons, circular sections (such as sectors and zones), convex and even starlike bodies with C^2 boundary [12, 13, 20]. Admissible meshes with near optimal cardinality can be constructed on planar convex bodies, and on fat subanalytic compacts; cf. [13, 17].

3 Numerical results

A number of theoretical and computational results have pointed out in the last years that WAMs are relevant structures for multivariate polynomial approximation.

Let us term $\mathcal{L}_{\mathcal{A}_n}$ the projection operator $C(K) \to \mathbb{P}_n^d(K)$ defined by polynomial least squares on a WAM, and $I_{\mathcal{F}_n}$ the projection operator defined by interpolation on Fekete points of degree n extracted from a WAM (Fekete points are points that maximize the absolute value of the Vandermonde determinant). Concerning their operator norms with respect to $\|\cdot\|_K$, in [8] it is proved that

$$\|\mathcal{L}_{\mathcal{A}_n}\| \lessapprox C(\mathcal{A}_n)\sqrt{\mathrm{card}(\mathcal{A}_n)}\,,\;\; \|I_{\mathcal{F}_n}\| \le C(\mathcal{A}_n)\,\dim(\mathbb{P}_n^d(K))\,, \tag{10}$$

which show that WAMs with slowly increasing constants $C(\mathcal{A}_n)$ and cardinalities, are sets of choice for multivariate polynomial approximation. Weakly admissible and even admissible meshes with $\mathcal{O}(n^d)$ cardinality are known to exist for several d-dimensional compacts, and in some cases are also easily computable; cf., e.g., [2, 5, 7, 13, 20].

The extraction of Fekete points from a WAM is a NP-hard problem, but two greedy algorithms, resting on basic matrix factorization methods, can be successfully adopted. Working on rectangular Vandermonde matrices in a suitable polynomial basis, by QR factorization with pivoting one computes the so-called "approximate Fekete points". On the other hand, LU factorization with pivoting allows to compute the so-called "discrete Leja sequences"; cf. [3, 18].

Both these families show good interpolation properties; moreover, it has been proved that they behave asymptotically as the "true" Fekete points, namely that the associated discrete measure converges to the pluripotential equilibrium measure of the compact [1, 2, 3].

On the other hand, all the numerical tests have shown that bounds (10) for the projection operators are by large overestimates of the actual norms (even using the approximate sets), cf., e.g., [2, 5].

We present some numerical results concerning the present 3-dimensional framework; the corresponding set of Matlab functions and demos can be downloaded from [11]. In Figures 1 and 3, we plot the WAM and the approximate Fekete points extracted from them, for a pyramid with quadrangular base, a truncated cone with circular base, and a portion of a torus with circular cross-section. The 3-dimensional WAMs have been constructed as in Theorem 1, starting from the 2-dimensional meshes studied in [5, 12], which have $\mathrm{card}(\mathcal{A}_n) \approx n^2$ and $C(\mathcal{A}_n) \approx (\frac{2}{\pi}\log n)^2$. Notice that the mesh points (and thus the approximate Fekete points) lie on a pencil of segments in the conical instances, and on a bundle of parallel circular arcs in the toroidal instance.

In Figures 2 and 4 the norms of the interpolation (left) and least squares (right) operators are shown (they have been evaluated numerically on a finer control WAM of degree $4n$). Such norms turn out to be much lower than the bounds (10).

However, it is not always possible to use a theoretical WAM for practical computations, especially in the present 3-dimensional instances, when the 2-dimensional meshes on Ω are already huge at moderate degree. For example, if Ω is a polygon, then by triangulation and finite union we can obtain a WAM that has constant $C(\mathcal{A}_n) \approx (\frac{2}{\pi}\log n)^2$, but with cardinality $\mathrm{card}(\mathcal{A}_n) \approx mn^2$, where m is the number of sides, which would led to a WAM for the cone or the solid of rotation with $\mathrm{card}(\mathcal{B}_n) \approx mn^3$. Already for moderate values of m and n, finding a WAM with lower cardinality could become essential to manage computational complexity and memory requirements.

In order to reduce the sampling cardinality, we could use the (until now only experimental) observation that the approximate Fekete points of degree $2n$ for a d-dimensional compact K, say $\tilde{\mathcal{F}}_{2n}$, present low norms in least squares approximation of degree n, and have clearly cardinality equal to $\dim(\mathbb{P}_{2n}^d(K))$ (irrespectively of the geometry of the compact). In Figure 5, we show for illustration the WAM and the approximate Fekete points of degree 10 on a regular decagon, and the norms of the least squares operator for degree n on the approximate Fekete points $\tilde{\mathcal{F}}_{2n}$, $n = 1, \ldots, 16$.

A possible theoretical interpretation of such a behavior could come from an open problem stated in [1]: *for every (L-regular) compact set in $K \subset \mathbb{C}^d$, does there exist $c = c(K) > 1$ such that Fekete arrays of degree cn form an admissible mesh for*

K? Notice that the fact that univariate Chebyshev-like algebraic and trigonometric interpolation sets of degree cn, $c > 1$, form an admissible mesh, is well-known; cf., e.g., [6, 13, 20]. The notion of L-regularity arises in pluripotential theory, and has strong connections with multivariate polynomial approximation. In the real case, a sufficient condition for L-regularity is for example that the compact domain is fat (i.e., $\overline{K^o} = K$), and subanalytic [15], which is essentially equivalent to the property that K is a finite union of images of cubes by open analytic mappings; we refer the reader to [16] for a survey on pluripotential theory and subanalytic geometry. In the present framework, it is not difficult to show that a cone or a solid of rotation are fat and subanalytic, and thus L-regular, whenever the planar compact set Ω is fat and subanalytic as a subset of the two real dimensional affine plane in which it lies. This is the case in all of our examples.

In general, given a $\mathbb{P}_n^d(K)$-determining finite set $X_n \subset K$, the least squares operator is a projection operator on the polynomial space, that can be written as $\mathcal{L}_{X_n} f(x) = \sum_{\xi \in X_n} f(\xi)\,\phi_\xi(x)$, where $\{\phi_\xi\}$ is a suitable array of generators of $\mathbb{P}_n^d(K)$, from which we get $\|\mathcal{L}_{X_n}\| = \max_{x\in K} \sum_{\xi\in X_n} |\phi_\xi(x)|$ and $\|\mathcal{L}_{X_n} f\|_K \le \|\mathcal{L}_{X_n}\|\,\|f\|_{X_n}$; cf., e.g., [4].

Then, (1) holds for the planar compact Ω with $\mathcal{A}_n = X_n = \tilde{\mathcal{F}}_{2n}$, namely

$$\|p\|_\Omega \le \|\mathcal{L}_{\tilde{\mathcal{F}}_{2n}}\|\,\|p\|_{\tilde{\mathcal{F}}_{2n}}\,, \quad \forall p \in \mathbb{P}_n^d(\Omega)\,, \tag{11}$$

i.e., $\tilde{\mathcal{F}}_{2n}$ is a WAM, at least for the range of degrees that have been experimentally tested. This entails that the corresponding WAM of the cone with base Ω, or of the solid of rotation with cross-section Ω, say $\mathcal{B}_n$, will have $C(\mathcal{B}_n) = \mathcal{O}(\|\mathcal{L}_{\tilde{\mathcal{F}}_{2n}}\| \log n)$ and $\text{card}(\mathcal{B}_n) \approx 2n^3$ (irrespectively of the geometry of Ω). A further reduction of the sampling cardinality, which becomes $(2n+1)(2n+2)(2n+3)/6 \approx \frac{4}{3}n^3$, can be obtained resorting again to the approximate Fekete points of degree $2n$ extracted from $\mathcal{B}_{2n}$: a numerical test is shown in Figure 6.

Figure 1: WAMs (asterisks) and approximate Fekete points (small circles) of degree $n = 4$ for a pyramid with quadrangular base, and for a standard truncated cone.

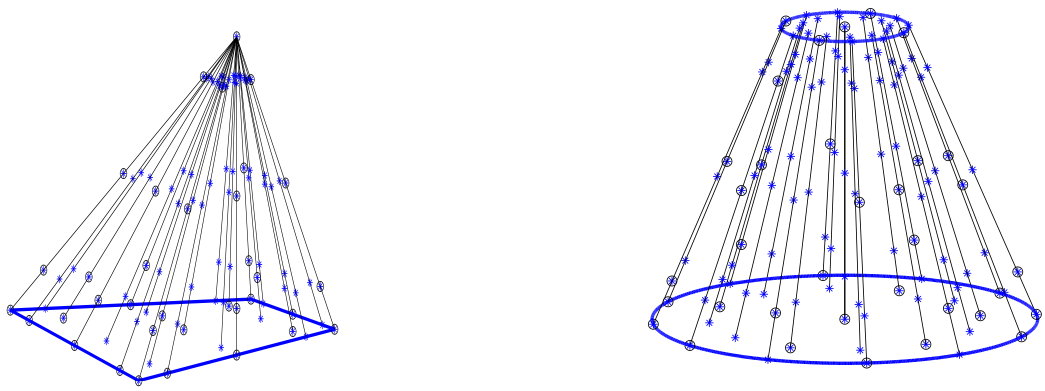

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Figure 2: Lebesgue constants of the approximate Fekete points (left) and norms of the least squares approximation operators (right), for the WAMs of the pyramid (small triangles) and truncated cone (small circles) in Figure 1, at degree $n = 1, \ldots, 20$.

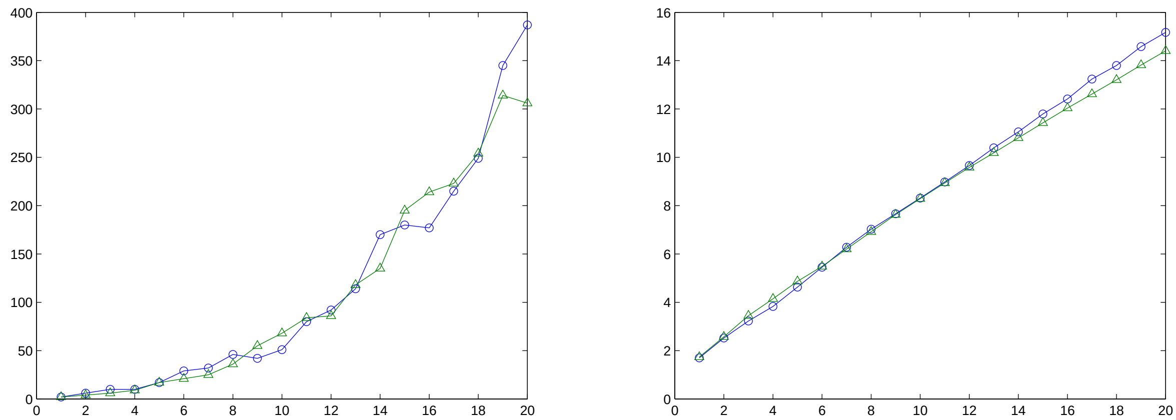

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Figure 3: WAM (asterisks) and approximate Fekete points (small circles) of degree $n = 3$ for a portion of torus corresponding to a rotation of a disk by an angle $2\pi/3$.

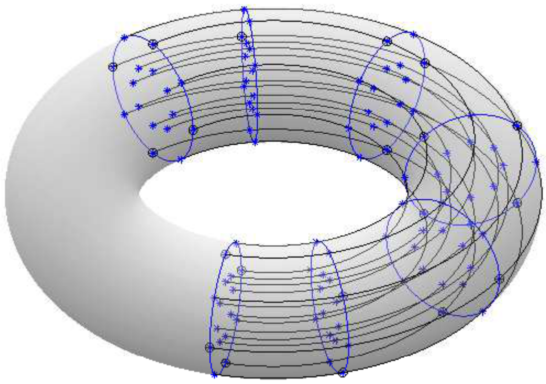


Figure 4: Lebesgue constants of the approximate Fekete points (left) and norms of the least squares approximation operators (right), for the WAMs of a portion of torus with circular section (small circles) and square section (small squares), at degree $n = 1, \ldots, 20$.

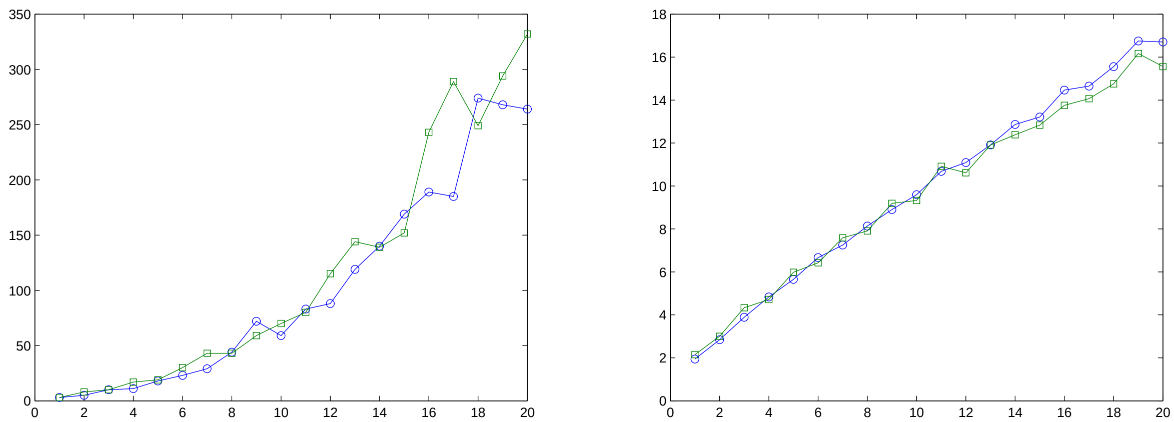
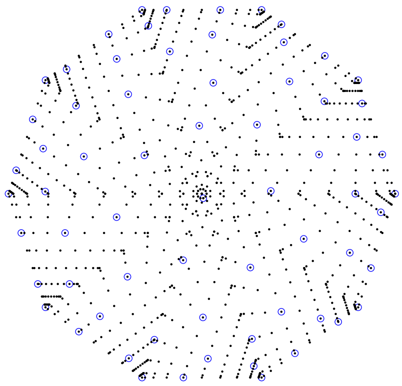


Figure 5: Left: WAM (dots) and corresponding approximate Fekete points (small circles) of degree 10 on a regular decagon. Right: norm of the least squares projection operator on approximate Fekete points of degree $2n$, $n = 1, \dots, 16$.



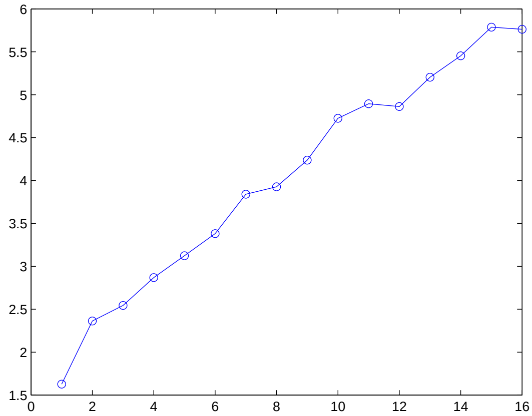


Figure 6: Left: WAM (asterisks) and corresponding approximate Fekete points of degree 4 (small circles) on a pyramid with decagonal base (the mesh of the base is given by approximate Fekete points of degree 8). Right: norm of the least squares projection operator on the pyramid's approximate Fekete points of degree $2n$, $n = 1, \dots, 16$.