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On effective transversely isotropic elasticity tensors based on Frobenius and L_2 operator norms

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Abstract

A generally anisotropic elasticity tensor, which might be obtained from physical measurements, can be approximated by a tensor belonging to a particular material-symmetry class; we refer to such a tensor as the effective tensor. The effective tensor is the closest to the generally anisotropic tensor among the tensors of that symmetry class. The concept of closeness is formalized in the notion of norm. Herein, we compare the effective tensors belonging to the transversely isotropic class and obtained using two different norms: the Frobenius norm and the L_2 operator norm. We compare distributions of the effective elasticity parameters and symmetry-axis orientations for both the error-free case and the case of the generally anisotropic tensor subject to errors.

1 Introduction

A Hookean solid is defined by the elasticity tensor, $c_{ijk\ell}$, which stems from the constitutive relation that relates linearly the stress, σ_{ij}, and strain, $\varepsilon_{k\ell}$, tensors,

$$\sigma_{ij} = \sum_{k,\ell=1}^{3} c_{ijk\ell}\varepsilon_{k\ell}\,, \qquad i,j \in \{1,2,3\}\,. \tag{1}$$

The elasticity tensor is assumed to satisfy the index symmetries, $c_{ijk\ell} = c_{jik\ell} = c_{k\ell ij}$, and be positive-definite; these requirements reduce the number of independent components to twenty-one and, hence, allow us to represent equation (1) as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\sigma_{23} \\ \sqrt{2}\sigma_{13} \\ \sqrt{2}\sigma_{12} \end{bmatrix} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & \sqrt{2}c_{1123} & \sqrt{2}c_{1113} & \sqrt{2}c_{1112} \\ c_{1122} & c_{2222} & c_{2233} & \sqrt{2}c_{2223} & \sqrt{2}c_{2213} & \sqrt{2}c_{2212} \\ c_{1133} & c_{2233} & c_{3333} & \sqrt{2}c_{3323} & \sqrt{2}c_{3313} & \sqrt{2}c_{3312} \\ \sqrt{2}c_{1123} & \sqrt{2}c_{2223} & \sqrt{2}c_{3323} & 2c_{2323} & 2c_{2313} & 2c_{2312} \\ \sqrt{2}c_{1113} & \sqrt{2}c_{2213} & \sqrt{2}c_{3313} & 2c_{2313} & 2c_{1313} & 2c_{1312} \\ \sqrt{2}c_{1112} & \sqrt{2}c_{2212} & \sqrt{2}c_{3312} & 2c_{2312} & 2c_{1312} & 2c_{1212} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \sqrt{2}\varepsilon_{23} \\ \sqrt{2}\varepsilon_{13} \\ \sqrt{2}\varepsilon_{12} \end{bmatrix}. \tag{2}$$

As shown by several researchers—among them Forte and Vianello [6] and Bóna et al. [1]—the elasticity tensor belongs to one of eight material-symmetry classes, from general anisotropy to isotropy. It is common to approximate a generally anisotropic tensor by its closest counterpart that belongs to particular symmetry classes. Such a tensor is referred to as the effective tensor of that class. There might be several motivations for such an approximation; in view of uncertainties of $c_{ijk\ell}$, we might require a model with fewer parameters; also, a particular symmetry might allow us to recognize properties of the material represented by the elasticity tensor. However, depending on the applied norm, we obtain different effective tensors of a given class.

In this paper, we consider the Frobenius norm and the operator norm induced by the L_2 norm on the spaces of the elasticity tensor in expression (1) to examine the transversely isotropic effective tensors, whose generic form is

$$\mathbf{c}^{\mathrm{TI}} = \begin{bmatrix} c^{\mathrm{TI}}_{1111} & c^{\mathrm{TI}}_{1122} & c^{\mathrm{TI}}_{1133} & 0 & 0 & 0 \\ c^{\mathrm{TI}}_{1122} & c^{\mathrm{TI}}_{1111} & c^{\mathrm{TI}}_{1133} & 0 & 0 & 0 \\ c^{\mathrm{TI}}_{1133} & c^{\mathrm{TI}}_{1133} & c^{\mathrm{TI}}_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2c^{\mathrm{TI}}_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c^{\mathrm{TI}}_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & c^{\mathrm{TI}}_{1111} - c^{\mathrm{TI}}_{1122} \end{bmatrix}. \tag{3}$$

In particular, we examine the effect of errors in the components of the generally anisotropic tensor on the parameters and orientations of its transversely isotropic effective tensors.

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2 Norms and distances for effective tensors

The Frobenius norm treats the square matrix in expression (2) as a Euclidean vector; it is the square root of the sum of squared components, $c_{ijk\ell}$. This norm exhibits the rotational invariance, which—in general—is convenient to study anisotropy; it is used commonly in such studies (e.g., Kochetov and Slawinski [8], Danek et al. [4]).

As discussed by Bos and Slawinski [2], another matrix norm that exhibits the rotational invariance is the induced Euclidean operator norm,

$$\|A\|_2 := \max_{\|x\|_2=1} \|Ax\|_2 ,$$

which—for $A = A^t$, as is the case of the square matrix, C, in expression (2)—becomes

$$\|A\|_2 := \max\{|\lambda| \, : \, \lambda \text{ an eigenvalue of } A\} ; \tag{4}$$

we refer to this norm as the operator norm.

For the elasticity tensor, $\mathbf{c}$, we let $\lambda_1, \ldots, \lambda_6$ be the eigenvalues of the corresponding matrix C; then, we write

$$\|\mathbf{c}\|_F := \sqrt{\sum_{i,j,k,\ell=1}^{3} c_{ijk\ell}^2} = \sqrt{\lambda_1^2 + \cdots + \lambda_6^2} , \tag{5}$$

and

$$\|\mathbf{c}\|_2 := \max\{|\lambda_i| \, : i = 1, \ldots, 6\} , \tag{6}$$

which are the Frobenius and operator norms, respectively.

The former corresponds to the quadratic average of the norm of the stress over strains whose norm is unity: $\sqrt{6 \operatorname{avg}_{\|\varepsilon\|=1} \|C\varepsilon\|^2} = \|\mathbf{c}\|_F$. The latter is the maximum of that quantity: $\max_{\|\varepsilon\|=1} \|C\varepsilon\| = \|\mathbf{c}\|_2$.

In view of expression (1), one could consider an operator norm induced by various norms for the stress and strain tensors, which are second-rank tensors; for example, one could consider the matrix operator L_2 norm or indeed another L_p norm. However, since the resulting norm is not the operator L_2 norm of the matrix representation, C, of tensor $\mathbf{c}$ (L. Bos, *pers.comm.*, 2013), we do not consider it in this paper. Herein, the norm of the stress and strain tensors is defined as the Euclidean length of their vectorial representations, as in expression (2).

The Frobenius-norm effective tensor, $\hat{\mathbf{c}}$, relative to the fixed orientation is the orthogonal projection of the generally anisotropic tensor, $\mathbf{c}$, in the sense of the Frobenius norm, on the linear space containing all tensors of a given symmetry. This projection is the average given by

$$\hat{\mathbf{c}} := \int\limits_{G^{\mathrm{sym}}(q)} (g \circ \mathbf{c}) \, \mathrm{d}\mu(g) , \tag{7}$$

where the integration is over the symmetry group, G^{sym}, whose elements are g, with respect to the invariant measure, μ, normalized so that $\mu(G^{\mathrm{sym}}) = 1$, as described by Gazis et al. [7]; q denotes the dependence of the value of integral (7) on the relative orientations of $\mathbf{c}$ and $\hat{\mathbf{c}}$. This integral reduces to a finite sum for the classes whose symmetry groups are finite, which are all classes except isotropy and transverse isotropy. As shown by Gazis et al. [7], projection (7) ensures that a positive-definite tensor is projected to another positive-definite tensor, as required by Hookean solids.

As shown by Moakher and Norris [9] and Bucataru and Slawinski [3]—in a fixed orientation of the rotation-symmetry axis that coincides with the x_3-axis of the coordinate system—the Frobenius-norm effective transversely isotropic tensor derived from expression (7) has the form of tensor (3) with components given by

$$\hat{c}_{1111}^{\mathrm{TI}} = \frac{1}{8}\left(3c_{1111} + 3c_{2222} + 2c_{1122} + 4c_{1212}\right) , \tag{8}$$

$$\hat{c}_{1122}^{\mathrm{TI}} = \frac{1}{8}\left(c_{1111} + c_{2222} + 6c_{1122} - 4c_{1212}\right) , \tag{9}$$

$$\hat{c}_{1133}^{\mathrm{TI}} = \frac{1}{2}\left(c_{1133} + c_{2233}\right) , \tag{10}$$

$$\hat{c}_{3333}^{\mathrm{TI}} = c_{3333} , \tag{11}$$

$$\hat{c}_{2323}^{\mathrm{TI}} = \frac{1}{2}\left(c_{2323} + c_{1313}\right) , \tag{12}$$

where $c_{ijk\ell}$ are the components of the generally anisotropic tensor. No analytic form of the operator-norm effective tensor is known.

In general, the distance between the generally anisotropic tensor and its effective counterpart, $\mathbf{c}_{\mathrm{eff}}$, expressed in the same orientation of the coordinate system, is

$$d(q) = \left\| \mathbf{c} - \mathbf{c}_{\mathrm{eff}}(q) \right\| , \tag{13}$$

where q denotes the orientation dependence. For the Frobenius case, $\|\,\|_F$, where the norm is given by expression (5), average (7) is tantamount to the orthogonal projection, and in view of that projection, $\mathbf{c}-\hat{\mathbf{c}}$ and $\hat{\mathbf{c}}$ are normal to one another; hence, we can write

$$d_F^2(q) = \|\,\mathbf{c}\,\|_F^2 - \|\,\hat{\mathbf{c}}(q)\,\|_F^2\,. \tag{14}$$

No such simplification is possible for the operator norm due to the absence of the concept of orthogonality, and angle, in general, since, herein, the operator norm is not an inner-product norm. Thus,

$$d_2(q) = \left\|\,\mathbf{c} - \tilde{\mathbf{c}}(q)\,\right\|_2\,, \tag{15}$$

where $\tilde{\mathbf{c}}$ is the operator-norm effective tensor. In the absence of analytic expressions, we compute the components of $\tilde{\mathbf{c}}$ numerically, which contributes to a difference in a computational difficulty between the Frobenius-norm and the operator-norm approaches.

In general—since expression (13) is orientation-dependent—to obtain the effective tensors, we must find the absolute minima of expressions (14) and (15), which are the minima under all orientations of the rotation-symmetry axis. This is a highly nonlinear problem, which we address with a global optimization method called particle-swarm optimization (PSO), discussed by Poli et al. [10], and by invoking quaternions, as used by Kochetov and Slawinski [8] and Danek et al. [4].

For the Frobenius case, the search for the absolute minimum of expression (14) is achieved by expressing $\mathbf{c}$ in all orientations of the coordinate system and, for each orientation, calculating $\hat{\mathbf{c}}$ whose components are stated in expressions (8)–(12). Since $\|\mathbf{c}\|$ is invariant under coordinate transformations, it suffices to maximize $\|\hat{\mathbf{c}}\|$ to minimize expression (14). This is a two-variable problem, whose variables are the angles that define the orientation of the rotation-symmetry axis of a transversely isotropic tensor.

Since no analytic form akin to expressions (8)–(12) is known for the operator effective tensor, we must search numerically for the corresponding $\tilde{\mathbf{c}}$; this effective tensor must have a form of tensor (3), and be such that the maximum eigenvalue of the difference between $\mathbf{c}$ and $\tilde{\mathbf{c}}$ is minimized; we must perform minimization under all orientations of the rotation-symmetry axis. This is a seven-variable problem, whose variables are the two angles that describe the orientation of the axis and the five elasticity parameters of the corresponding effective tensor, $\tilde{\mathbf{c}}$. Furthermore, one must verify that each candidate for the effective tensor is positive-definite, as required for a Hookean solid, since—unlike for the Frobenius-effective tensor—this requirement is not intrinsically satisfied for the operator effective tensor, as discussed by Bos and Slawinski [2].

3 Numerical example

3.1 Effective tensors

To examine differences between the Frobenius norm and the operator norm, let us compare the transversely isotropic effective tensors obtained using these norms. To do so, we use the generally anisotropic elasticity tensor obtained from seismic measurements by Dewangan and Grechka [5],

$$\mathbf{c} = \begin{bmatrix} 7.8195 & 3.4495 & 2.5667 & \sqrt{2}(0.1374) & \sqrt{2}(0.0558) & \sqrt{2}(0.1239) \\ 3.4495 & 8.1284 & 2.3589 & \sqrt{2}(0.0812) & \sqrt{2}(0.0735) & \sqrt{2}(0.1692) \\ 2.5667 & 2.3589 & 7.0908 & \sqrt{2}(-0.0092) & \sqrt{2}(0.0286) & \sqrt{2}(0.1655) \\ \sqrt{2}(0.1374) & \sqrt{2}(0.0812) & \sqrt{2}(-0.0092) & 2(1.6636) & 2(-0.0787) & 2(0.1053) \\ \sqrt{2}(0.0558) & \sqrt{2}(0.0735) & \sqrt{2}(0.0286) & 2(-0.0787) & 2(2.0660) & 2(-0.1517) \\ \sqrt{2}(0.1239) & \sqrt{2}(0.1692) & \sqrt{2}(0.1655) & 2(0.1053) & 2(-0.1517) & 2(2.4270) \end{bmatrix}, \tag{16}$$

whose entries of this matrix are the density-scaled elasticity parameters; their units are $\mathrm{km}^2/\mathrm{s}^2$. In other words, the Hookean solid in question is completely described by expression (16). The corresponding standard deviations are

$$S = \pm\begin{bmatrix} 0.1656 & 0.1122 & 0.1216 & \sqrt{2}(0.1176) & \sqrt{2}(0.0774) & \sqrt{2}(0.0741) \\ 0.1122 & 0.1862 & 0.1551 & \sqrt{2}(0.0797) & \sqrt{2}(0.1137) & \sqrt{2}(0.0832) \\ 0.1216 & 0.1551 & 0.1439 & \sqrt{2}(0.0856) & \sqrt{2}(0.0662) & \sqrt{2}(0.1010) \\ \sqrt{2}(0.1176) & \sqrt{2}(0.0797) & \sqrt{2}(0.0856) & 2(0.0714) & 2(0.0496) & 2(0.0542) \\ \sqrt{2}(0.0774) & \sqrt{2}(0.1137) & \sqrt{2}(0.0662) & 2(0.0496) & 2(0.0626) & 2(0.0621) \\ \sqrt{2}(0.0741) & \sqrt{2}(0.0832) & \sqrt{2}(0.1010) & 2(0.0542) & 2(0.0621) & 2(0.0802) \end{bmatrix}. \tag{17}$$

As shown by Danek et al. [4], the effective tensor—in the Frobenius sense—is at the distance of $1.0733\,\mathrm{km}^2/\mathrm{s}^2$ from tensor (16), and is given by

$$\hat{\mathbf{c}} = \begin{bmatrix} 8.0641 & 3.3720 & 2.4588 & 0 & 0 & 0 \\ 3.3720 & 8.0640 & 2.4588 & 0 & 0 & 0 \\ 2.4588 & 2.4588 & 7.0817 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1.8625) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1.8625) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(2.3460) \end{bmatrix}, \tag{18}$$

with orientation determined by

$$q = \pm(0.993636 - 0.010573\,\mathbf{i} + 0.027463\,\mathbf{j} + 0.108726\,\mathbf{k})\,, \tag{19}$$

where q is a unit quaternion, $q = a + b\,\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where $a^2+b^2+c^2+d^2=1$, $\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{i}\mathbf{j}\mathbf{k}=-1$, and a, b, c, d are real numbers.

Recall that a unit quaternion gives rise to a rotation in the space of purely imaginary quaternions, p, which are quaternions whose $a=0$. This is a consequence of the fact that $p \mapsto q\,p\,\bar{q}$, where $\bar{q} \equiv a - b\,\mathbf{i} - c\mathbf{j} - d\mathbf{k}$ is the conjugate of q, maps purely imaginary quaternions to purely imaginary quaternions, which we can view as vectors relative to basis $\{\mathbf{i}\,\mathbf{j}\,\mathbf{k}\}$. Also, since $\|p\| = 1$, it follows that $\|qp\bar{q}\| = \|p\|$, which means that it is a norm-preserving transformation. Thus, we can write $p' = q\,p\,\bar{q}$, where p' is p rotated by

$$\begin{bmatrix} a^2+b^2-c^2-d^2 & -2ad+2bc & 2ac+2bd \\ 2ad+2bc & a^2-b^2+c^2-d^2 & -2ab+2cd \\ -2ac+2bd & 2ab+2cd & a^2-b^2-c^2+d^2 \end{bmatrix}, \tag{20}$$

which is the rotation matrix corresponding to q.

The effective tensor—in the operator sense—is at the distance of $0.6022\,\mathrm{km}^2/\mathrm{s}^2$, and is given by

$$\tilde{\mathbf{c}} = \begin{bmatrix} 8.1154 & 3.1557 & 2.6736 & 0 & 0 & 0 \\ 3.1557 & 8.1154 & 2.6736 & 0 & 0 & 0 \\ 2.6736 & 2.673646 & 6.9239 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1.8630) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1.8630) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(2.4798) \end{bmatrix}, \tag{21}$$

with orientation determined by

$$q = \pm(0.989244 + 0.009249\,\mathbf{i} - 0.004980\,\mathbf{j} - 0.145900\,\mathbf{k}). \tag{22}$$

3.2 Physical quantities

Let us consider differences in physical quantities resulting from the choice of a norm. Since the velocity of the P wave along the rotation-symmetry axis is $\sqrt{c_{3333}}$, we see that the estimates based on tensors (18) and (21) are $2.84\,\mathrm{km/s}$ and $2.85\,\mathrm{km/s}$, respectively; in the context of seismic measurements, this might be a negligible difference. Notably, this difference is smaller than for the effective isotropic tensors, where—as shown by Bos and Slawinski [2]—the velocities are $2.71\,\mathrm{km/s}$ and $2.76\,\mathrm{km/s}$, respectively. The smaller difference might be due to a larger number of parameters: two-parameter isotropic model versus seven-parameter transversely isotropic model. A choice of a model must involve a consideration of errors associated with $c_{ijk\ell}$, which we examine below.

For a transversely isotropic medium, the velocity of the SH wave exhibits an elliptical dependence whose semiaxis parallel to the rotation-symmetry axis has the magnitude of $\sqrt{c_{2323}}$, and the other semiaxis has the magnitude of $\sqrt{c_{1212}}$. The ellipticity is symptomatic of the strength of anisotropy. For tensor (18), $(c_{1212}-c_{2323})/c_{2323}$ is 0.26, and for tensor (21), it is 0.33; thus, even in the context of a limited accuracy of seismic measurements, this might be a nonnegligible discrepancy between information provided by the two norms.

A comparison of effective tensors is not complete without the examination of their orientations. Converting expressions (19) and (22) to the Euler angles—which, even though less convenient for computations than quaternions, might be easier to visualize—we see that the orientations of their rotation-symmetry axes differ by $\Delta\theta = 0.30^\circ$ and $\Delta\phi = 2.75^\circ$, where θ and ϕ are the azimuth and tilt respectively. Again, in the context of seismic measurements—where the axis orientation is interpreted as the orientation of subsurface layers—this might be a negligible difference.

3.3 Perturbations

To examine the effect of errors on the distribution of results, let us consider the perturbation method, whose results are illustrated in Figures 1 and 2. Using errors (17) and the Monte-Carlo method, we perturb tensor (16) a thousand times, and—using either norm—find the corresponding effective tensor.

The computational effort required to find the operator effective tensors is significantly greater than to find the Frobenius effective tensors. There are two main reasons for the increased effort.

First, as discussed in Section 2, not only the orientation—as is the case for the Frobenius norm—but also the elasticity parameters are the subject of search. Thus, the dimensionality of the problem is increased.

Second, the maximum eigenvalue has to be computed for every solution candidate. To do so, we apply a modification of the PSO algorithm used by Danek et al. [4] by incorporating the singular value decomposition (SVD) into the target function. Fortunately, the well-known flexibility of PSO allows us to obtain results effectively enough to consider a perturbation analysis.

Examining Figures 1 and 2, we see that histograms—in particular, θ and ϕ—obtained for the operator norm are wider. This is a result of the optimization process, where only the maximum energy in directions of the eigenvectors is taken into consideration; thus, global minima are found within wider basins of attraction. Also, unlike in Figure 1, histograms in Figure 2 exhibit a lack of alignment between the maxima of empirical distributions and results obtained for the error-free data. The understanding of this shift requires further investigation.

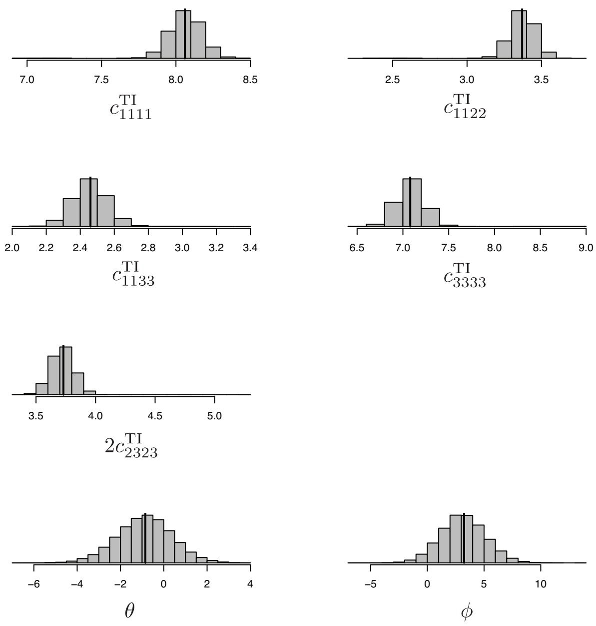

Figure 1: Histograms of the five density-scaled elasticity parameters of the Frobenius effective transversely isotropic tensors and of the two Euler angles describing the azimuth, θ , and tilt, ϕ , of their rotation-symmetry axes. These histograms are obtained from realizations of tensor (16) subject to errors (17); black vertical lines correspond to the error-free case. [4]

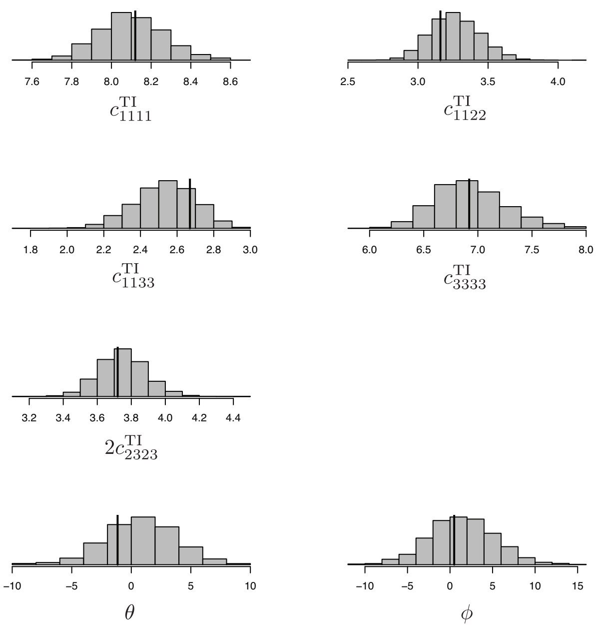

Figure 2: Histograms of the five density-scaled elasticity parameters of the operator effective transversely isotropic tensors and of the two Euler angles describing the azimuth, θ , and tilt, ϕ , of their rotation-symmetry axes. These histograms are obtained from realizations of tensor (16) subject to errors (17); black vertical lines correspond to the error-free case.

4 Conclusions

The choice of norm has an impact on the resulting effective tensor. Differences between the operator-norm effective tensor and the Frobenius-norm effective tensor are due to several reasons; notably, the operator norm takes into account the index symmetry of the tensor, $c_{ijk\ell} = c_{k\ell ij}$, whereas the Frobenius norm does not.

In general, an optimization of a target function can be based on any valid criterion. Herein, both approaches can be used for perturbation methods to evaluate qualitatively the effect of errors. However, until a rigorous statistical analysis is performed, such a distinction between the norms remains qualitative. A quantitative statistical analysis can be done using maximum-likelihood estimators, which require error-weighted norms. Such an approach is under investigation.

Be that as it may, it is important to recognize that, while tensor (16) is a mathematical analogy for physical properties of a material measured by Dewangan and Grechka [5], tensors (18) and (21) are approximations of that tensor. Two distinct mathematical approximations referring to the same physical object remind us of an obvious, yet commonly neglected, distinction between the physical world and the realm of mathematics.

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