Dolomites Research Notes on Approximation
Volume 17 · 2024 · Pages 59–63

On the log-convexity of a Bernstein-like polynomials sequence

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Dedicated to Professor Ioan Raşa on the occasion of his 70th birthday

Abstract

We prove that the sequence of the sum of the squares of the Bernstein polynomials is pointwise log-convex. There are given two proofs of this result: one by relating our sequence to the Legendre polynomials sequence and one by induction. I know of this problem from Professor Ioan Rasa, Cluj-Napoca. This work was presented at the International Conference on Approximation Theory and its Applications, Sibiu, 2022, dedicated to the scientific work of Professor Ioan Rasa on the occasion of his 70th anniversary.

1 Introduction

Let $\{x_n\}_{n\geq 0}$ be a sequence of real nonnegative numbers. The sequence is called **log-convex** (resp. **log-concave**) if ${x_k}^2 \leq x_{k-1}x_{k+1}$ (resp. $x_{k-1}x_{k+1} \leq {x_k}^2$) for any k. Log-convexity implies **convexity**, defined by $2x_k \leq x_{k-1}+x_{k+1}$. On the other hand, **concavity**, given by $x_{k-1}+x_{k+1} \leq 2x_k$, for any k implies log-concavity. A function $f : I \longrightarrow \mathbb{R}$ (where $I \subset \mathbb{R}$ is an interval) is said to be **completely monotonic** if f has derivatives of all orders and satisfies

$$0 \leq (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} f(x), k = 0, 1, 2, ..., x \in I$$

If

$$0 \leq \frac{\mathrm{d}^k}{\mathrm{d}x^k} f(x), k = 0, 1, 2, ..., x \in I$$

then the function f is called **absolutely monotonic**. If f is a completely monotonic function then the sequence $c_n = f(n)$ is **a completely monotonic sequence** i.e.

$$(-1)^k(\Delta^k c)_n \geq 0$$

where

$$(\Delta^0 c)_n = c_n, (\Delta c)_n = c_{n+1} - c_n, \Delta^{k+1} = \Delta(\Delta^k)$$

A Hausdorff moment sequence is one of the form

$$h_n = \int_0^1 t^n d\,\nu(t)$$

where ν is a positive measure on $[0, 1]$. Hausdorff moment sequences were characterized as completely monotonic sequences by Hausdorff in the fundamental paper [4]. These sequences are bounded Stieltjes moment sequences i.e. of the form

$$s_n = \int_0^\infty t^n d\,\nu(t)$$

for a positive measure ν on $[0, \infty)$
Let q be an indeterminate. Given two real polynomials $f(q)$ and $g(q)$, write $f(q) \preceq g(q)$ if and only if $g(q) - f(q)$ has only nonnegative coefficients as polynomial in q. A sequence of polynomials $\{P_n(q)\}$, with $n \geq 0$, is called **q-log-convex** if

$$(P_n(q))^2 \preceq P_{n-1}(q)P_{n+1}(q)$$

for all $n \geq 1$, that is $P_{n-1}(q)P_{n+1}(q) - (P_n(q))^2$ has only nonnegative coefficients as polynomial in q. Obviously, if the polynomial sequence $\{P_n(q)\}$, with $n \geq 0$, is q-log-convex, then for each fixed positive number q, the sequence $\{P_n(q)\}$ is log-convex, or the polynomial sequence is **pointwise log-convex**. See [5]
There have been written a lot of papers concerned with the log-concavity of sequences (see the article of L. Liu and Y. Wang [5] for some (fairly) recent developments and an extended bibliography related to the subject). However, the systematic study of the log-convexity of polynomial sequences has more to show. Log-convexity is, in a sense, more challenging property than log-concavity.

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We denote by $S_n(x)$ the sum of the squared Bernstein polynomials,

$$S_n(x) = \sum_{k=0}^{n} (B_k(n,x))^2, x \in [0,1]$$

where

$$B_k(n,x) = \binom{n}{k} x^k (1-x)^{n-k}$$

In 2014 Gonska and al. [2] conjectured that the **function** S_n was convex on $[0,1]$, for any natural number n. In the same year Gavrea and Ivan [3] and Nikolov [6] proved the conjecture true. Also in 2014, Rasa conjectured that S_n was even log-convex, conjecture that has been proved true by Rasa himself in [7] in 2018. The proof is based on some properties of the Legendre polynomials. In 2019 in [8] Rasa gave a simplified proof by using functional equations satisfied by S_n. Finally, in 2020 Alzer [1] made use of an elegant representation of S_n, obtained by Rasa (in a slightly different form also by Gavrea and Nikolov) to prove the log-convexity of S_n. In fact he proved that the **function** S_n is completely monotonic on $[0,1/2]$ and absolutely monotonic on $[1/2,1]$. As the author noticed, this (in many cases) stronger result does not imply the log-convexity since S_n is not completely monotonic on $(0,\infty)$.

In this paper we prove the **log-convexity in the other argument** i.e. we present proofs of the fact that the **sequence** $\{S_n(x)\}_{n\geq 0}$ is log-convex for any $x \in [0,1]$. This distinction between the log-convexity "in x" and the log-convexity "in n" is the main feature of this paper. The first proof is based on some representation of $S_n(x)$ in terms of Legendre polynomials and the second one is a direct proof, by induction, that does not use any representation, coming from Legendre polynomials or not.

2 Log-convexity and Legendre polynomials

The main result in this section is the following representation:
Theorem 2.1.

$$S_n(x) = (-2x+1)^n P_n\left(\frac{-2x^2+2x-1}{2x-1}\right)$$

where

$$P_n(t) = \frac{\sum_{k=0}^{n} \left(\binom{n}{k}\right)^2 (t-1)^{n-k}(t+1)^k}{2^n}$$

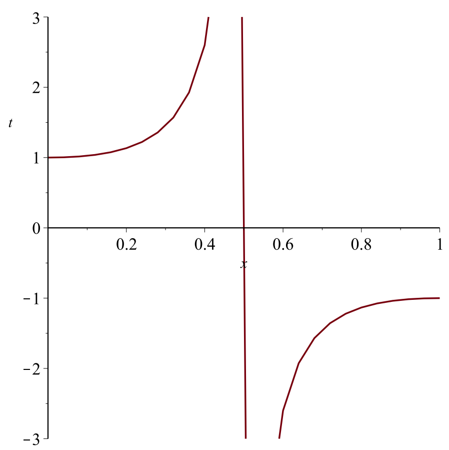
are the Legendre poynomials.
The proof is straightforward and consists in replacing in the above representation of Legendre polynomials t with

$$t = \frac{-2x^2+2x-1}{2x-1}$$

followed by $t-1 = 2x^2(-2x+1)^{-1}$ and $t+1 = 2(x-1)^2(-2x+1)^{-1}$.

Remark 2.1. The representation above is also obtained in [7] and [8].

The following graph gives us the dependence of t by x.



One can see that, for $x \in [0,1/2)$, $t \in [1,\infty)$ and for $x \in (1/2,1]$, t is in the interval $(-\infty,-1)$. Let's make the following notation

$$\Delta_n(x) = (S_n(x))^2 - S_{n-1}(x)S_{n+1}(x).$$

We have to prove that $\Delta_n(x) \leq 0$ for any $x \in [0,1]$.
For $t \in [1,\infty)$ a proof will be given, for $x = 1/2$ a direct computation will reach to the same conclusion and for $t \in (-\infty,-1)$ the symmetry relation $P_n(-t) = (-1)^n P_n(t)$ will be used.
The main result above directly leads to:
Theorem 2.2.

$$\Delta_n(x) = (-2x+1)^{2n}\left((P_n(t))^2 - P_{n-1}(t)P_{n+1}(t)\right)$$

with the above notations.
It follows that $\Delta_n(x)$ has the same sign with

$$(P_n(t))^2 - P_{n-1}(t)P_{n+1}(t).$$

Proof. The Turan inequality:

$$0 \geq (P_n(x))^2 - P_{n-1}(x)P_{n+1}(x)$$

for $x \in (-1,1)$ is well known (see [13]).
On the other hand there is no real x such that $t \in (-1,1)$.

1. Now, for $t \in [1,\infty)$ the Legendre polynomials have the integral representation:

$$P_n(t) = \frac{\int_0^{\pi}\left(t - \cos(\theta)\sqrt{t^2-1}\right)^n \, d\theta}{\pi}.$$

It is a simple consequence of the Cauchy-Buneacovski-Schwartz inequality the fact that any sequence, as above, defined by $p_n = \int_a^b (f(\theta))^n \, d\theta$, with positive f, is log-convex.
Remark 2.2. The integral representation of Legendre polynomials together with the log-convexity of the sequences of the form

$$p_n = \int_a^b (f(\theta))^n$$

provide a very simple, almost straightforward proof of the log-convexity of the **sequence** S_n (see Remark 2 bellow).

2. For $x = 1/2$, $S_n(1/2) = \frac{\Gamma(n+1/2)}{\sqrt{\pi}\Gamma(n+1)}$.

Indeed, the sum $\sum_{k=0}^{n} \binom{n}{k}^2$ is a Vandermonde binomial sum type and, after it is written in the form $\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$, one can see that its value is $\binom{2n}{n}$ or, for better simplifications, $\frac{\Gamma(1+2n)}{(\Gamma(n+1))^2}$.
Therefore, after some simplification computations,

$$\Delta_n(1/2) = -\frac{4^n(n+1)(\Gamma(n+1/2))^2}{\pi(-1+2n)(\Gamma(n+2))^2}$$

hence $\Delta_n(1/2) \leq 0$.

3. For $t \in (-\infty,-1)$, $(P_n(-t))^2 - P_{n-1}(-t)P_{n+1}(-t) = (P_n(t))^2 - P_{n-1}(t)P_{n+1}(t)$, so, again, $\Delta_n(x) \leq 0$.

In conclusion, $\Delta_n(x) \leq 0$ for any $x \in [0,1]$ that is, the sequence of polynomials is pointwise log-convex.

Remarks:
1. The sequence of polynomials is NOT x-log-convex in the terminology of L. Liu and Y. Wang from [5].
In our case, the 16 degree polynomial $\Delta_8(x) \leq 0$ has nonnegative coefficients and negative coefficients as well, as one can see below, where this polynomial has been expanded.
$-140x^{16} + 1120x^{15} - 4256x^{14} + 10192x^{13} - 17228x^{12} + 21832x^{11} - 21532x^{10} + 16976x^9 - 10906x^8 + 5744x^7 - 2440x^6 + 800x^5 - 188x^4 + 28x^3 - 2x^2$.
2. Using the formulas:
$S_n(x) = (-2x+1)^n P_n\left(\frac{-2x^2+2x-1}{2x-1}\right)$ and $P_n(t) = \frac{\int_0^{\pi}\left(t-\cos(\theta)\sqrt{t^2-1}\right)^n d\theta}{\pi}$ we obtain the following integral representation where $x \in [0,1/2)$.

$$S_n(x) = \frac{\int_0^{\pi}\left(2x^2 - 2x + 1 + \left(2x^2 - 2x\right)\cos(y)\right)^n \, dy}{\pi}.$$

For $x = \frac{1}{2}$ we have
$S_n(1/2) = \frac{\Gamma(n+1/2)}{\sqrt{\pi}\Gamma(n+1)}$ that agrees with $\frac{\int_0^{\pi}(1/2 - 1/2\cos(y))^n \, dy}{\pi}$.
This tells us that the above integral representation is valid, for $x \in [0,1/2]$. Therefore, the sequence $S_n(x)$ is, at least for any $x \in [0,1/2]$, **very close to be** a Hausdorff moment sequence, that is much more than log-convex!

3 Log-convexity by induction

The main result of this section is presented below.
Theorem 3.1. The sequence of polynomials $\{S_n(x)\}_{n\geq 0}$ satisfies the following three terms recurrence.

$$(n+1)S_{n+1}(x) = \left(2x^2-2x+1\right)(2n+1)S_n(x) - (2x-1)^2 nS_{n-1}(x)$$

for $n \geq 1$. The first terms are $S_0(x) = 1$ and $S_1(x) = 2x^2-2x+1$.

To prove this just notice that if $t = \frac{-2x^2+2x-1}{2x-1}$ then $t(1-2x) = 2x^2-2x+1$. Using the representation of $S_n(x)$ in terms of $P_n(t)$ from the previous section, the recurrence we have to prove comes to the well known:

$$(n+1)P_{n+1}(t) = (2n+1)tP_n(t) - nP_{n-1}(t)$$

and the initial terms are the same.

Remark. This recurrence is of the type

$$a_n(x)S_{n+1}(x) = b_n(x)S_n(x) - c_n(x)S_{n-1}(x)$$

for $n \geq 1$, where the sequences: S_n, a_n, b_n, c_n are all positive (see [5]).

Theorem 3.2. The sequence $\{S_n(x)\}_{n\geq 0}$ is log-convex for any $x \neq 1/2$.

Proof. Let's make the (standard) notation $q_n(x) = \frac{S_{n+1}(x)}{S_n(x)}$ and notice that all we have to do is to prove that the sequence $q_n(x)$ is increasing for any x. In what follows we'll drop x from $q_n(x)$, and from all other sequences, and we'll simply write q_n. The recurrence satisfied by q_n, coming from the recurrence of S_n, is

$$q_{n+1} = \frac{\left(2x^2-2x+1\right)(2n+3)}{n+2} - \frac{(1-2x)^2(n+1)}{(n+2)q_n}$$

with $q_0 = 2x^2-2x+1$.
The argument for the induction step comes from the result below:

$$q_{n+1} - q_n = \frac{[\left(2x^2-2x+1\right)q_n - (1-2x)^2]}{(n+2)(n+1)q_n} + \frac{(1-2x)^2 n(q_n - q_{n-1})}{(n+1)q_n q_{n-1}}.$$

One can see that if

$$0 \leq \left(2x^2-2x+1\right)q_n - (1-2x)^2$$

then, from $0 \leq q_n - q_{n-1}$ it follows that $0 \leq q_{n+1} - q_n$, meaning that the sequence q_n is increasing. We'll prove, by induction, that

$$0 \leq \left(2x^2-2x+1\right)q_n - (1-2x)^2$$

for any $n \geq 0$.

For $n = 0$ we'll have to prove that $0 \leq \left(2x^2-2x+1\right)^2 - (1-2x)^2$ that is obvious. Now, let's suppose that $0 \leq \left(2x^2-2x+1\right)q_n - (1-2x)^2$ or, in a more useful form,

$$-\frac{2x^2-2x+1}{(1-2x)^2} \leq -q_n{}^{-1}.$$

From the recurrence satisfied by q_n we have

$$\left(2x^2-2x+1\right)q_{n+1} = \frac{2x^2-2x+1}{n+2}[\left(2x^2-2x+1\right)(2n+3) - \frac{(1-2x)^2(n+1)}{q_n}].$$

Using the induction hypothesis, the above bracket is greater than

$$\left(2x^2-2x+1\right)(2n+3) - \frac{(1-2x)^2(n+1)\left(2x^2-2x+1\right)}{(1-2x)^2} = \left(2x^2-2x+1\right)(n+2)$$

so, the left hand side from the above equality, containing q_{n+1}, is greater than $\left(2x^2-2x+1\right)^2$. Hence,

$$\left(2x^2-2x+1\right)^2 - (1-2x)^2 \leq \left(2x^2-2x+1\right)q_{n+1} - (1-2x)^2.$$

In conclusion, $4x^2(x-1)^2 \leq \left(2x^2-2x+1\right)q_{n+1} - (1-2x)^2$ and the proof is completed.
Remarks:
1. In this second proof is easier to see that the sequence $S_n(x)$ is log-convex for **any** real x different from $1/2$.

2. For $x = 1/2$ we can not just replace x with $1/2$ in the three terms recurrence above. This gives us $nS_n(1/2) = 1/2\,(-1+2\,n)\,S_{n-1}(1/2)$, with $S_1(1/2) = 1/2$, whereas the definition of the polynomial sequence from the beginning gives us the recurrence $nS_n(1/2) = (-1+2\,n)\,S_{n-1}(1/2)$, $S_1(1/2) = 1$. In this case is easy to prove, by induction, that $S_n(1/2)$ is log-convex.

3. For any natural number n the function, in x, $S_n(x)$ is one solution of some Heun differential equation, see [9].

4. Following an idea of G. Polya, G. Szego proved the point wise log-concavity of some polynomial sequences showing that their generating function is of certain type. These are precisely entire functions that are limits of polynomials with real roots only. On the other hand if the generating function of a real numbers sequence is a Pick function then the sequence is log-convex (see [10]). In fact, these sequences are even Hausdorff moment sequences! In [5] there are given some conditions for the point wise log-convexity but not an explicit generating function for log-convex polynomial sequences; such an analogous to the Polya functions but for point wise log-convex polynomial sequences, even for (numerical) sequences, would have been of great interest.

5. This problem is similar, from one point of view, with the following problem, coming also from Professor Ioan Rasa: prove that the sequence

$$c(n,k) = \binom{2n}{k}^{-2} \sum_{j=0}^{k} \binom{n}{j}^2 \binom{n}{k-j}^2$$

is convex in k for any n. Numerical computations seem to lead to the fact that this sequence is, in fact, a Hausdorff moment sequence, or completely monotone sequence. Both sequences are "probabilistic" sequences i.e. the sums of the **unsquared** terms is 1. Indeed, $\sum_{k=0}^{n} \binom{n}{k}(1-x)^{n-k}x^k = 1$ for any x in $[0,1]$ and, similarly, $\binom{2n}{k}^{-1} \sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-j} = 1$ for any n. So, the sequences are convex combinations of a particular type! The general problem would be: suppose the positive terms sequence $\{a\,(n,k)\}_{k\geq 0}$ satisfies $\sum_{k=0}^{n} a\,(n,k) = 1$ for any n. In what conditions the sequence $A_n = \sum_{k=0}^{n} (a\,(n,k))^2$ would be: convex, log-convex, Hausdorff moment sequence? This can be formulated in probabilistic terms.

Acknowledgements.
I thank Professor I. Rasa for the valuable remarks and advices made after the careful readings of this paper.

References

[1] H. Alzer, *Remarks on a Convexity Theorem of Rasa*, Results Math (2020) 75: 29.

[2] Gonska, H., Rasa, I., Rusu, M.D, *Chebyshev-Grüss-type inequalities via discrete oscillations.*, Bul. Acad. Stiinte Repub. Mold. Mat. 1, 63-89 (2014).

[3] Gavrea I., Ivan M., *On a conjecture concerning the sum of the squared Bernstein polynomials*, Appl. Math. Comput. 241, 70-74, (2014).

[4] F. Hausdorff, "*Summationsmethoden und momentfolgen I*", Mathematische Zeitschrift, vol. 9, no. 1-2, pp. 74-109, 1921.

[5] L. Liu, Y. Wang, *On the log-convexity of combinatorial sequences*, Advances in Applied Mathematics 39(4) (2007), 453-476.

[6] Nikolov, G., *Inequalities for ultraspherical polynomials. Proof of a conjecture of I. Rasa*, J. Math. Anal. Appl. 418, 852-860, (2014).

[7] Rasa,I., *Convexity Properties of Some Entropies*, Results Math (2018) 73: 105.

[8] Rasa,I., *Convexity Properties of Some Entropies II*, Results Math (2019) 74: 154.

[9] Rasa, I., *Special functions associated to positive linear operators*, arXiv:1409.1015v2 [Math.CA] 2014.

[10] O.Roth, S. Ruscheweyh, L. Salinas, *A note on generating functions for Hausdorff moment sequences* , Proc. Amer. Math. Soc., 136(9), 3171-3176.

[11] G. Szego, *On an inequality of P. Turan concerning Legendre polynomials*, Bull. Amer. Math. Soc. 54(1948), 401-405.

[12] G. Szego, *Orthogonal Polynomials*, A.M.S. Colloquium Publications, vol, 23, N.Y., 1939.

[13] H. Skovgaard, *On the inequalities of the Turan type*, MATH. SCAND. 2(1954), 65-73.

[14] http://mathworld.wolfram.com/Legendre.html

[15] https://en.wikipedia.org/wiki/Legendre polynomials