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Brass-Stancu-Kantorovich operators on a Hypercube*

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Dedicated to Professor Ioan Raşa on the occasion of his 70th birthday

Abstract

We deal with multivariate Brass-Stancu-Kantorovich operators depending on a non-negative integer parameter and defined on the space of all Lebesgue integrable functions on a unit hypercube. We prove L^p-approximation and provide estimates for the L^p-norm of the error of approximation in terms of a multivariate averaged modulus of continuity and of the corresponding L^p-modulus.

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1 Introduction and Historical Notes

The fundamental functions of the well-known Bernstein operators are defined by

$$p_{n,k}(x)=\begin{cases}\binom{n}{k}x^k(1-x)^{n-k}; & 0\le k\le n\\ 0; & k<0 \text{ or } k>n\end{cases},\quad x\in[0,1]. \tag{1}$$

In [23], using a probabilistic method, Stancu generalized Bernstein's fundamental functions as

$$w_{n,k,r}(x):=\begin{cases}(1-x)\,p_{n-r,k}(x); & 0\le k<r\\ (1-x)\,p_{n-r,k}(x)+x\,p_{n-r,k-r}(x); & r\le k\le n-r\\ x\,p_{n-r,k-r}(x); & n-r<k\le n\end{cases},\quad x\in[0,1], \tag{2}$$

where r is a non-negative integer parameter, n is any natural number such that $n>2r$, for which each $p_{n-r,k}$ is given by (1), and therefore, constructed and studied Bernstein-type positive linear operators as

$$L_{n,r}(f;x):=\sum_{k=0}^{n}w_{n,k,r}(x)f\left(\frac{k}{n}\right),\quad x\in[0,1], \tag{3}$$

for $f\in C[0,1]$. In doing so Stancu was guided by an article of Brass [8]. This is further discussed by Gonska [11]. Among others, estimates in terms of the second order modulus of smoothness are given there for continuous functions.

It is clear that for $x\in[0,1]$ Stancu's fundamental functions in (2) satisfy

$$w_{n,k,r}(x)\ge 0 \text{ and } \sum_{k=0}^{n}w_{n,k,r}(x)=1,$$

hence the operators $L_{n,r}$ can be expressed as

$$L_{n,r}(f;x):=\sum_{k=0}^{n-r}p_{n-r,k}(x)\left[(1-x)f\left(\frac{k}{n}\right)+xf\left(\frac{k+r}{n}\right)\right], \tag{4}$$

are defined for $n\ge r$ and satisfy the end point interpolation $L_{n,r}(f;0)=f(0)$, $L_{n,r}(f;1)=f(1)$. It is thus justified to call the $L_{n,r}$ Brass-Stancu-Bernstein (BSB) operators.

In [24] Stancu gave uniform convergence $\lim_{n\to\infty}L_{n,r}(f)=f$ on $[0,1]$ for $f\in C[0,1]$ and presented an expression for the remainder $R_{n,r}(f;x)$ of the approximation formula $f(x)=L_{n,r}(f;x)+R_{n,r}(f;x)$ by means of second order divided differences and also obtained an integral representation for the remainder. Moreover, the author estimated the order of approximation by the operators $L_{n,r}(f)$ via the classical modulus of continuity. He also studied the spectral properties of $L_{n,r}$.

In the cases $r=0$ and $r=1$, the operators $L_{n,r}$ reduce to the classical Bernstein operators B_n, i.e.,

$$B_n(f;x)=\sum_{k=0}^{n}p_{n,k}(x)f\left(\frac{k}{n}\right).$$

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What also has to be mentioned: Stancu himself in his 1983 paper observed that "we can optimize the error bound of the approximation of the function f by means of $L_{n,r}f$ if we take $r=0$ or $r=1$, when the operator $L_{n,r}$ reduces to Bernstein's." So there is a shortcoming.

Since Bernstein polynomials are not appropriate for approximation of discontinuous functions (see [14, Section 1.9]), by replacing the point evaluations $f\left(\frac{k}{n}\right)$ with the integral means over small intervals around the knots $\frac{k}{n}$, Kantorovich [12] generalized the Bernstein operators as

$$K_n(f;x)=\sum_{k=0}^{n} p_{n,k}(x)(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt,\quad x\in[0,1],\ n\in\mathbb{N}, \tag{5}$$

for Lebesgue integrable functions f on $[0,1]$.

On p. 239 of his mathematical memoirs [13] Kantorovich writes: "While I was waiting for a student who was late, I was looking over vol. XIII of Fundamenta Math. and saw in it a note from the Moscow Mathematician Khlodovskii related to Bernstein polynomials. In it I first caught sight of Bernstein polynomials, which he proposed in 1912 for an elementary proof of the well known Weierstrass theorem ... I at once wondered if it is not possible in these polynomials to change the values of the function at certain points into the more stable average of the function in the corresponding interval. It turned out that this was possible, and the polynomials could be written in such a form not only for a continuous function but also for any Lebesgue-summable function."

Lorentz [14] proved that $\lim\limits_{n\to\infty}\|K_n(f)-f\|_p=0, f\in L^p[0,1],\ 1\le p<\infty$.

There are a lot of articles dealing with classical Kantorovich operators, and, in particular, their degree of approximation and the importance of second order moduli of different types. See, e.g., the work of Berens and DeVore [5], [6], Swetits and Wood [25] and Gonska and Zhou [10]. It is beyond the scope of this note to further discuss this matter. As further work on the classical case here we only mention the 1976 work of Müller [16], Maier [15], and Altomare et al. [1], see also the references therein.

Similarly to Kantorovich operators Bodur et al. [7] constructed a Kantorovich type modification of BSB operators as

$$K_{n,r}(f;x):=\sum_{k=0}^{n} w_{n,k,r}(x)\left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt\right),\quad x\in[0,1], \tag{6}$$

for $f\in L^1[0,1]$, where r is a non-negative integer parameter, n is a natural number such that $n>2r$ and $w_{n,k,r}(x)$ are given by (2). And, it was shown that *If* $f\in L^p[0,1],\ 1\le p<\infty$, *then* $\lim\limits_{n\to\infty}\left\|K_{n,r}(f)-f\right\|_p=0$. In addition, it was obtained that each $K_{n,r}$ is variation detracting as well [7]. Throughout the paper, we shall call the operators $K_{n,r}$ given by (6) "Brass-Stancu-Kantorovich", BSK operators.

Note that from the definition of $w_{n,k,r}$, $K_{n,r}(f;x)$ can be expressed as

$$K_{n,r}(f;x)=\sum_{k=0}^{n-r} p_{n-r,k}(x)(n+1)\left[(1-x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt+x\int_{\frac{k+r}{n+1}}^{\frac{k+r+1}{n+1}} f(t)\,dt\right]$$

and in the cases $r=0$ and $r=1$ they reduce to the Kantorovich operators; $K_{n,0}=K_{n,1}=K_n$ given by (5). Again they are defined for all $n\ge r$.

MULTIVARIATE SITUATION

Some work has been done in the multivariate setting for BSB and BSK operators. For the standard simplex this was done, e.g., by Yang, Xiong and Cao [27] and Cao [9], For example, Cao proved that multivariate Stancu operators preserve the properties of multivariate moduli of continuity and obtained the rate of convergence with the help of Ditzian-Totik's modulus of continuity.

In this work, motivated by the work Altomare et al. [3], we deal with a multivariate extension of the BSK operators on a d-dimensional unit hypercube and we study L^p -approximation by these operators. For the rate of convergence we provide an estimate in terms of the so called first order multivariate τ-modulus, a quantity coming from the Bulgarian school of Approximation Theory. Also, inspired by Müller's approach in [17], we give estimates for differentiable functions and such in terms of the L^p-modulus of smoothness, using properties of the τ-modulus. Here the work of Quak [20], [21] was helpful.

2 Preliminaries

Consider the space $\mathbb{R}^d$, $d\in\mathbb{N}$. Let $\|\mathbf{x}\|_\infty$ denote the max-norm of a point $\mathbf{x}=(x_1,\dots,x_d)\in\mathbb{R}^d$;

$$\|\mathbf{x}\|_\infty:=\|\mathbf{x}\|_{\max}=\max_{i\in\{1,\dots,d\}}|x_i|$$

and let $\mathbf{1}$ denote the constant function $\mathbf{1}:\mathbb{R}^d\to\mathbb{R}$ such that $\mathbf{1}(\mathbf{x})=1$ for $\mathbf{x}\in\mathbb{R}^d$. And, for each $j=1,\dots,d$, let

$$pr_j:\mathbb{R}^d\to\mathbb{R}$$

stand for the jth coordinate function defined for $\mathbf{x}\in\mathbb{R}^d$ by

$$pr_j(\mathbf{x})=x_j.$$

Definition 2.1. A multi-index is a d-tuple $\alpha = (\alpha_1, \ldots, \alpha_d)$ of non-negative integers. Its norm (length) is the quantity

$$|\alpha| = \sum_{i=1}^{d} \alpha_i.$$

The differential operator D^α is defined by

$$D^\alpha f = D_1^{\alpha_1} \cdots D_d^{\alpha_d} f,$$

where D_i, $i = 1, \ldots, d$, is the corresponding partial derivative operator (see [4, p. 335]).

Throughout the paper $Q_d := [0,1]^d$, $d \in \mathbb{N}$, will denote the d-dimensional unit hypercube and we consider the space

$$L^p(Q_d) = \{f : Q_d \to \mathbb{R} \mid f \text{ } p\text{-integrable on } Q_d\},\ 1 \le p < \infty,$$

with the standard norm $\|.\|_p$. Recall the following definition of the L^p-modulus of smoothness of first order:

Definition 2.2. Let $f \in L^p(Q_d)$, $1 \le p < \infty$, $\mathbf{h} \in \mathbb{R}^d$ and $\delta > 0$. The modulus of smoothness of the first order for the function f and step δ in L^p-norm is given by

$$\omega_1(f;\delta)_p = \sup_{0 < \|\mathbf{h}\|_\infty \le \delta} \left(\int_{Q_d} |f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})|^p \, d\mathbf{x} \right)^{1/p}$$

if $\mathbf{x}, \mathbf{x}+\mathbf{h} \in Q_d$ [21].

Let $M(Q_d) := \{f \mid f$ bounded and measurable on $Q_d\}$. Below, we present the concept of the first order averaged modulus of smoothness.

Definition 2.3. *Let $f \in M(Q_d)$, $\mathbf{h} \in \mathbb{R}^d$ and $\delta > 0$. The multivariate averaged modulus of smoothness, or τ-modulus, of the first order for function f and step δ in L^p-norm is given by*

$$\tau_1(f,\delta)_p := \|\omega_1(f,.;\delta)\|_p,\ 1 \le p < \infty,$$

where

$$\omega_1(f,\mathbf{x};\delta) = \sup\left\{|f(\mathbf{t}+\mathbf{h}) - f(\mathbf{t})| : \mathbf{t}, \mathbf{t}+\mathbf{h} \in Q_d,\ \|\mathbf{t}-\mathbf{x}\|_\infty \le \tfrac{\delta}{2}, \|\mathbf{t}+\mathbf{h}-\mathbf{x}\|_\infty \le \tfrac{\delta}{2}\right\}$$

is the multivariate local modulus of smoothness of first order for the function f at the point $\mathbf{x} \in Q_d$ and for step δ. [21].

For our future purposes, we need the following properties of the first order multivariate averaged modulus of smoothness: For $f \in M(Q_d)$, $1 \le p < \infty$ and $\delta, \lambda, \gamma \in \mathbb{R}^+$, there hold

$\tau_1)$ $\tau_1(f,\delta)_p \le \tau_1(f,\lambda)_p$ for $0 < \delta \le \lambda$,

$\tau_2)$ $\tau_1(f,\lambda\delta)_p \le (2\lfloor\lambda\rfloor + 2)^{d+1} \tau_1(f,\delta)_p$, where $\lfloor\lambda\rfloor$ is the greatest integer less than or equal to λ,

$\tau_3)$ $\tau_1(f,\delta)_p \le 2 \sum_{|\alpha| \ge 1} \delta^{|\alpha|} \|D^\alpha f\|_p$, $\alpha_i = 0$ or 1, if $D^\alpha f \in L^p(Q_d)$ for all multi-indices α with $|\alpha| \ge 1$ and $\alpha_i = 0$ or 1 (see [19] or [21]).

For a detailed knowledge concerning averaged modulus of smoothness, we refer to the book of Sendov and Popov [22].

Now, consider the Sobolev space $W_1^p(Q_d)$ of functions $f \in L^p(Q_d)$, $1 \le p < \infty$, with (distributional) derivatives $D^\alpha f$ belonging to $L^p(Q_d)$, where $|\alpha| \le 1$, with seminorm

$$|f|_{W_1^p} = \sum_{|\alpha|=1} \|D^\alpha f\|_p$$

(see [4, p. 336]). Recall that for all $f \in L^p(Q_d)$ the K-functional, in L^p-norm, is defined as

$$K_{1,p}(f;t) := \inf\left\{\|f-g\|_p + t|g|_{W_1^p} : g \in W_1^p(Q_d)\right\} \quad (t > 0). \tag{7}$$

$K_{1,p}(f;t)$ is equivalent with the usual first order modulus of smoothness of f, $\omega_1(f;t)_p$; namely, there are positive constants c_1 and c_2 such that

$$c_1 K_{1,p}(f;t) \le \omega_1(f;t)_p \le c_2 K_{1,p}(f;t) \quad (t > 0) \tag{8}$$

holds for all $f \in L^p(Q_d)$ (see [4, Formula 4.42 in p. 341]).

The following result due to Quak [21] is an upper estimate for the L^p-norm of the approximation error by the multivariate positive linear operators in terms of the first order averaged modulus of smoothness. Note that this idea was used first by Popov for the univariate case in [18].

Theorem 2.1. *Let $L : M(Q_d) \to M(Q_d)$ be a positive linear operator that preserves the constants. Then for every $f \in M(Q_d)$ and $1 \le p < \infty$, the following estimate holds:*

$$\|L(f) - f\|_p \le C\tau_1\left(f, \sqrt[2d]{A}\right)_p,$$

where C is a positive constant and

$$A := \sup\left\{L\left((pr_i \circ \psi_{\mathbf{x}})^2; \mathbf{x}\right) : i = 1, \ldots, d,\ \mathbf{x} \in Q_d\right\},$$

in which $\psi_{\mathbf{x}}(\mathbf{y}) := \mathbf{y} - \mathbf{x}$ for fixed $\mathbf{x} \in Q_d$ and for every $\mathbf{y} \in Q_d$ and $A \le 1$ [21].

3 Multivariate BSK Operators

In this section, motivated by Altomare et al. ([1], [3]), we consider the multivariate extension of BSK operators on $L^p(Q_d)$ and study approximation properties of these operators in L^p-norm. We investigate the rate of the convergence in terms of first order τ-modulus and the usual L^p-modulus of smoothness of first order.

Let r be a given non-negative integer. For any $n \in \mathbb{N}$ such that $n > 2r$, $\mathbf{k} = (k_1, \ldots, k_d) \in \{0, \ldots, n\}^d$ and $\mathbf{x} = (x_1, \ldots, x_d) \in Q_d$, we set

$$w_{n,\mathbf{k},r}(\mathbf{x}) := \prod_{i=1}^{d} w_{n,k_i,r}(x_i), \tag{9}$$

where, $w_{n,k_i,r}(x_i)$ is Stancu's fundamental function given by (2), written for each $i = 1, \ldots, d$, $0 \le k_i \le n$ and $x_i \in [0,1]$. Thus, for $\mathbf{x} \in Q_d$, we have

$$w_{n,\mathbf{k},r}(\mathbf{x}) \ge 0 \text{ and } \sum_{\mathbf{k}\in\{0,\ldots,n\}^d} w_{n,\mathbf{k},r}(\mathbf{x}) = 1. \tag{10}$$

For $f \in L^1(Q_d)$ and $\mathbf{x} = (x_1, \ldots, x_d) \in Q_d$ we consider the following multivariate extension of the BSK operators $K_{n,r}$ given by (6):

$$K_{n,r}^d(f;\mathbf{x}) = \sum_{k_1,\ldots,k_d=0}^{n} \prod_{i=1}^{d} w_{n,k_i,r}(x_i) \int_{Q_d} f\left(\frac{k_1+u_1}{n+1}, \ldots, \frac{k_d+u_d}{n+1}\right) du_1 \cdots du_d.$$

Notice that from (9), and denoting, as usual, any $f \in L^1(Q_d)$ of $\mathbf{x} = (x_1, \ldots, x_d) \in Q_d$ by $f(\mathbf{x}) = f(x_1, \ldots, x_d)$, we can express these operators in compact form as

$$K_{n,r}^d(f;\mathbf{x}) = \sum_{\mathbf{k}\in\{0,\ldots,n\}^d} w_{n,\mathbf{k},r}(\mathbf{x}) \int_{Q_d} f\left(\frac{\mathbf{k}+\mathbf{u}}{n+1}\right) d\mathbf{u}. \tag{11}$$

It is clear that multivariate BSK operators are positive and linear and the cases $r = 0$ and 1 give the multivariate Kantorovich operators on the hypercube Q_d, which can be captured from [1] as a special case.

Lemma 3.1. *For $\mathbf{x} \in Q_d$, we have*

$$\begin{aligned}
K_{n,r}^d(\mathbf{1};\mathbf{x}) &= 1,\\
K_{n,r}^d(pr_i;\mathbf{x}) &= \frac{n}{n+1}x_i + \frac{1}{2(n+1)},\\
K_{n,r}^d\left(pr_i^2;\mathbf{x}\right) &= \frac{n^2}{(n+1)^2}\left[x_i^2 + \left(1 + \frac{r(r-1)}{n}\right)\frac{x_i(1-x_i)}{n}\right]\\
&\quad + \frac{3nx_i+1}{3(n+1)^2},
\end{aligned}$$

for $i = 1, \ldots, d$.

Taking this lemma into consideration, by the well-known theorem of Volkov [26], we immediately get that

Theorem 3.2. *Let r be a non-negative fixed integer and $f \in C(Q_d)$. Then $\lim_{n\to\infty} K_{n,r}^d(f) = f$ uniformly on Q_d.*

Now, we need the following evaluations for the subsequent result: For $0 \le x_i \le 1$, $i = 1, \ldots, d$, we have

$$\begin{aligned}
\int_0^1 (1-x_i)\, p_{n-r,k_i}(x_i)\, dx_i &= \binom{n-r}{k_i} \int_0^1 x_i^{k_i}(1-x_i)^{n-r-k_i+1}\, dx_i\\
&= \frac{n-r-k_i+1}{(n-r+2)(n-r+1)}
\end{aligned}$$

when $0 \le k_i < r$ and

$$\begin{aligned}
\int_0^1 x_i p_{n-r,k_i-r}(x_i)\, dx_i &= \binom{n-r}{k_i-r} \int_0^1 x_i^{k_i-r+1}(1-x_i)^{n-k_i}\, dx_i\\
&= \frac{k_i-r+1}{(n-r+2)(n-r+1)}
\end{aligned}$$

when $n - r < k_i \le n$. Thus, from (1) and (2), it follows that

$$\int_0^1 w_{n,k_i,r}(x_i)dx_i = \begin{cases} \frac{n-r-k_i+1}{(n-r+2)(n-r+1)}; & 0 \le k_i < r\\ \frac{n-2r+2}{(n-r+2)(n-r+1)}; & r \le k_i \le n-r\\ \frac{k_i-r+1}{(n-r+2)(n-r+1)}; & n-r < k_i \le n \end{cases}. \tag{12}$$

Note that we have the following estimates:

$$
\begin{aligned}
n-r-k_i+1 &\leq n-r+1 \text{ when } 0 \leq k_i < r, \\
n-2r+2 &\leq n-r+1 \text{ when } r \leq k_i \leq n-r, \\
k_i-r+1 &\leq n-r+1 \text{ when } n-r < k_i \leq n
\end{aligned} \tag{13}
$$

for each $i = 1, \ldots, d$, where in the middle term, we have used the hypothesis $n > 2r$. Making use of (13), (12) and (9), we obtain

$$
\int_{Q_d} w_{n,\mathbf{k},r}(\mathbf{x})d\mathbf{x} = \prod_{i=1}^{d} \int_0^1 w_{n,k_i,r}(x_i)dx_i \leq \frac{1}{(n-r+2)^d}. \tag{14}
$$

L^p-approximation by the sequence of multivariate BSK operators is presented in the following theorem.

Theorem 3.3. *Let r be a non-negative fixed integer and $f \in L^p(Q_d)$, $1 \leq p < \infty$. Then $\lim_{n\to\infty} \left\| K_{n,r}^d(f) - f \right\|_p = 0$.*

Proof. Since the cases $r = 0$ and 1 correspond to the multivariate Kantorovich operators (see [1] or [3]), we consider only the cases $r > 1$, which is taken as fixed. From Theorem 3.2, we obtain that $\lim_{n\to\infty} \left\| K_{n,r}^d(f) - f \right\|_p = 0$ for any $f \in C(Q_d)$. Since $C(Q_d)$ is dense in $L^p(Q_d)$, denoting the norm of the operator $K_{n,r}^d$ acting on $L^p(Q_d)$ onto itself by $\left\| K_{n,r}^d \right\|$, it remains to show that there exists an M_r, where M_r is a positive constant that maybe depends on r, such that $\left\| K_{n,r}^d \right\| \leq M_r$ for all $n > 2r$. Now, as in [3, p.604], we adopt the notation

$$
Q_{n,\mathbf{k}} := \prod_{i=1}^{d} \left[\frac{k_i}{n+1}, \frac{k_i+1}{n+1} \right] \subset Q_d; \quad \bigcup_{\mathbf{k}\in\{0,\ldots,n\}^d} Q_{n,\mathbf{k}} = Q_d.
$$

Making use of the convexity of the function $\varphi(t) := |t|^p$, $t \in \mathbb{R}$, $1 \leq p < \infty$ (see, e.g., [2]), and (10), for every $f \in L^p(Q_d)$, $n > 2r$, and $\mathbf{x} \in Q_d$, we obtain

$$
\begin{aligned}
\left| K_{n,r}^d(f;\mathbf{x}) \right|^p &\leq \sum_{\mathbf{k}\in\{0,\ldots,n\}^d} w_{n,\mathbf{k},r}(\mathbf{x}) \int_{Q_d} \left| f\left(\frac{\mathbf{k}+\mathbf{u}}{n+1} \right) \right|^p d\mathbf{u} \\
&= \sum_{\mathbf{k}\in\{0,\ldots,n\}^d} w_{n,\mathbf{k},r}(\mathbf{x}) (n+1)^d \int_{Q_{n,\mathbf{k}}} |f(\mathbf{v})|^p d\mathbf{v}.
\end{aligned}
$$

Taking (14) into consideration, this yields

$$
\int_{Q_d} \left| K_{n,r}^d(f;\mathbf{x}) \right|^p d\mathbf{x} \leq \sum_{\mathbf{k}\in\{0,\ldots,n\}^d} \left(\frac{n+1}{n-r+2} \right)^d \int_{Q_{n,\mathbf{k}}} |f(\mathbf{v})|^p d\mathbf{v}.
$$

Since $\sup_{n>2r} \left(\frac{n+1}{n-r+2} \right)^d = \left(\frac{2r+2}{r+3} \right)^d := M_r$ for $r > 1$, where $1 < \frac{2r+2}{r+3} < 2$, we get

$$
\int_{Q_d} \left| K_{n,r}^d(f;\mathbf{x}) \right|^p d\mathbf{x} \leq M_r \int_{Qd} |f(\mathbf{v})|^p d\mathbf{v},
$$

which implies that $\left\| K_{n,r}^d(f) \right\|_p \leq M_r^{1/p} \|f\|_p$. Note that for the cases $r = 0$ and 1; we have $M_r = 1$ (see [3]). Therefore, the proof is completed. □

4 The rate of convergence

In [17], Müller studied L^p-approximation by the sequence of the Cheney-Sharma-Kantorovich operators (CSK). The author gave an estimate for this approximation in terms of the univariate τ-modulus and moreover, using some properties of the τ-modulus, he also obtained upper estimates for the L^p-norm of the error of approximation for first order differentiable functions as well as for continuous ones. In this part, we show that similar estimates can also be obtained for $\left\| K_{n,r}^d(f) - f \right\|_p$ in the multivariate setting. Our first result is an application of Quak's method in Theorem 2.1

Theorem 4.1. *Let r be a non-negative fixed integer, $f \in M(Q_d)$ and $1 \leq p < \infty$. Then*

$$
\left\| K_{n,r}^d(f) - f \right\|_p \leq C\tau_1 \left(f, \sqrt[2d]{\frac{3n+1+3r(r-1)}{12(n+1)^2}} \right)_p \tag{15}
$$

for all $n \in \mathbb{N}$ such that $n > 2r$, where the positive constant C does not depend on f.

Proof. According to Theorem 2.1; by taking $\psi_{\mathbf{x}}(\mathbf{y})=\mathbf{y}-\mathbf{x}$ for fixed $\mathbf{x}\in Q_d$ and for every $\mathbf{y}\in Q_d$, and defining

$$A_{n,r}:=\sup\left\{K_{n,r}^d\left((pr_i\circ\psi_{\mathbf{x}})^2;\mathbf{x}\right):i=1,\ldots,d,\ \mathbf{x}\in Q_d\right\},$$

where $(pr_i\circ\psi_{\mathbf{x}})^2=pr_i^2-2x_ipr_i+x_i^2\mathbf{1}$, $i=1,\ldots,d$, we get the following estimate

$$\left\|K_{n,r}^d(f)-f\right\|_p\leq C\tau_1\left(f;\sqrt[2d]{A_{n,r}}\right)$$

for any $f\in M(Q_d)$, under the condition that $A_{n,r}\leq 1$. Now, applying the operators $K_{n,r}^d$ and making use of Lemma 3.1, for every $i=1,\ldots,d$ and $\mathbf{x}\in Q_d$, we obtain

$$\begin{aligned}K_{n,r}^d\left((pr_i\circ\psi_{\mathbf{x}})^2;\mathbf{x}\right)&=\frac{n-1+r(r-1)}{(n+1)^2}x_i(1-x_i)+\frac{1}{3(n+1)^2}\\&\leq\frac{n-1+r(r-1)}{4(n+1)^2}+\frac{1}{3(n+1)^2}\\&=\frac{3n+1+3r(r-1)}{12(n+1)^2}\end{aligned}$$

for all $n\in\mathbb{N}$ such that $n>2r$, where $r\in\mathbb{N}\cup\{0\}$. Therefore, since we have $n\geq 2r+1$, we take $r\leq\frac{n-1}{2}$ and obtain that $A_{n,r}\leq\frac{3n+1+3r(r-1)}{12(n+1)^2}\leq 1$ is satisfied, which completes the proof. □

Now, making use of the properties $\tau 1)$-$\tau 3)$ of the multivariate first order τ-modulus, we obtain

Theorem 4.2. *Let r be a non-negative fixed integer, $f\in L^p(Q_d)$, $1\leq p<\infty$, and $D^\alpha f\in L^p(Q_d)$ for all multi-indices α with $|\alpha|\geq 1$, $\alpha_i=0$ or 1. Then*

$$\left\|K_{n,r}^d(f)-f\right\|_p\leq 2C_r\sum_{|\alpha|\geq 1}\left(\frac{1}{\sqrt[2d]{n+1}}\right)^{|\alpha|}\|D^\alpha f\|_p,$$

for all $n\in\mathbb{N}$ such that $n>2r$, where C_r is a positive constant depending on r.

Proof. Since $n>2r$, we immediately have $n+1\geq 2(r+1)$. Thus, the term appearing inside the $2d$th root in the formula (15) can be estimated, respectively, for $r>1$, and $r=0,1$, as

$$\begin{aligned}\frac{3n+1+3r(r-1)}{12(n+1)^2}&=\frac{3n+3+3r(r-1)-2}{12(n+1)^2}\\&=\frac{1}{n+1}\left[\frac{1}{4}+\frac{3r(r-1)-2}{12(n+1)}\right]\\&\leq\frac{1}{n+1}\left[\frac{1}{4}+\frac{3r(r-1)-2}{24(r+1)}\right]\\&=\frac{1}{n+1}\left[\frac{3r^2+3r+4}{24(r+1)}\right]\end{aligned}$$

and

$$\frac{3n+1}{12(n+1)^2}=\frac{1}{n+1}\frac{3n+1}{4(3n+3)}<\frac{1}{4(n+1)}.$$

Now, defining

$$B_r:=\begin{cases}\frac{3r^2+3r+4}{24(r+1)}; & r>1,\\ \frac{1}{4}; & r=0,1,\end{cases}$$

and making use of the properties $\tau_1)$-$\tau_3)$ of τ-modulus, from (15), we arrive at

$$\begin{aligned}\left\|K_{n,r}^d(f)-f\right\|_p&\leq C\tau_1\left(f,\sqrt[2d]{\frac{3n+1+3r(r-1)}{12(n+1)^2}}\right)_p\\&\leq C\tau_1\left(f,\sqrt[2d]{B_r}\frac{1}{\sqrt[2d]{n+1}}\right)_p\\&\leq C\left(2\left\lfloor\sqrt[2d]{B_r}\right\rfloor+2\right)^{d+1}\tau_1\left(f,\frac{1}{\sqrt[2d]{n+1}}\right)_p\\&\leq 2C_r\sum_{|\alpha|\geq 1}\left(\frac{1}{\sqrt[2d]{n+1}}\right)^{|\alpha|}\|D^\alpha f\|_p,\end{aligned}$$

where the positive constant C_r is defined as $C_r:=C\left(2\left\lfloor\sqrt[2d]{B_r}\right\rfloor+2\right)^{d+1}$. □

For non-differentiable functions we have the following estimate in terms of the first order modulus of smoothness, in L^p-norm.

Theorem 4.3. *Let r be a non-negative fixed integer and $f \in L^p(Q_d)$, $1 \le p < \infty$. Then*

$$\left\| K_{n,r}^d(f) - f \right\|_p \le c_2 C_{r,p} \omega_1 \left(f; \frac{1}{\sqrt[2d]{n+1}} \right)_p ,$$

where ω_1 is the first order multivariate modulus of smoothness of f and $C_{r,p}$ is a constant depending on r and p.

Proof. By Theorem 3.3, since $K_{n,r}^d$ is bounded, with $\left\| K_{n,r}^d \right\|_p \le M_r^{1/p}$, for all $n \in \mathbb{N}$ such that $n > 2r$, we have $\left\| K_{n,r}^d(g) - g \right\|_p \le \left(M_r^{1/p} + 1 \right) \|g\|_p$ for $g \in L^p(Q_d)$. Moreover, from Theorem 4.2, we can write

$$\left\| K_{n,r}^d(g) - g \right\|_p \le 2C_r \sum_{|\alpha| \ge 1} \left(\frac{1}{\sqrt[2d]{n+1}} \right)^{|\alpha|} \|D^\alpha g\|_p$$

for those g such that $D^\alpha g \in L^p(Q_d)$, for all multi-indices α with $|\alpha| \ge 1$ and $\alpha_i = 0$ or 1. Hence, for $f \in L^p(Q_d)$, it readily follows that

$$\begin{aligned} \left\| K_{n,r}^d(f) - f \right\|_p &\le \left\| K_{n,r}^d(f-g) - (f-g) \right\|_p + \left\| K_{n,r}^d(g) - g \right\|_p \\ &\le \left(M_r^{1/p} + 1 \right) \left\{ \|f-g\|_p + 2C_r \sum_{|\alpha| \ge 1} \left(\frac{1}{\sqrt[2d]{n+1}} \right)^{|\alpha|} \|D^\alpha g\|_p \right\}. \end{aligned}$$

Passing to the infimum for all $g \in W_1^p(Q_d)$ in the last formula, since the infimum of a superset does not exceed that of subset, we obtain

$$\begin{aligned} \left\| K_{n,r}^d(f) - f \right\|_p &\le \left(M_r^{1/p} + 1 \right) \inf \left\{ \|f-g\|_p + \frac{2C_r}{\sqrt[2d]{n+1}} \sum_{|\alpha|=1} \|D^\alpha g\|_p : g \in W_1^p(Q_d) \right\} \\ &= \left(M_r^{1/p} + 1 \right) \inf \left\{ \|f-g\|_p + \frac{2C_r}{\sqrt[2d]{n+1}} |g|_{W_1^p} : g \in W_1^p(Q_d) \right\} \\ &= \left(M_r^{1/p} + 1 \right) K_{1,p} \left(f; \frac{2C_r}{\sqrt[2d]{n+1}} \right), \end{aligned} \tag{16}$$

where $K_{1,p}$ is the K-functional given by (7). The proof follows from the equivalence (8) of the K-functional and the first order modulus of smoothness in L^p-norm and the non-decreasingness property of the modulus. Indeed, we get

$$\begin{aligned} K_{1,p} \left(f; \frac{2C_r}{\sqrt[2d]{n+1}} \right) &\le c_2 \omega_1 \left(f; \frac{2C_r}{\sqrt[2d]{n+1}} \right)_p \\ &\le c_2 (2C_r + 1) \omega_1 \left(f; \frac{1}{\sqrt[2d]{n+1}} \right)_p . \end{aligned} \tag{17}$$

Combining (17) with (16) and defining $C_{r,p} := \left(M_r^{1/p} + 1 \right)(2C_r + 1)$, where $M_r^{1/p}$ and C_r are the same as in Theorems 3.3 and 4.2, respectively, we obtain the desired result. □

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