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On some representation formulae for operator semigroups in terms of integrated means

Francesco Altomare[a] · Mirella Cappelletti Montano[b] · Vita Leonessa[c]

Dedicated to Professor Ioan Raşa on the occasion of his 70th birthday

Abstract

The aim of the paper is to develop some representation formulae for strongly continuous operator semigroups on Banach spaces, in terms of limits of integrated means with respect to some given family of probability Borel measures and other parameters.
The cases where these limits hold true pointwise or uniformly on compact subintervals are discussed separately. In order to face them different methods have been required: the former case has been studied by using purely functional-analytic methods, the latter one by involving methods arising from Approximation Theory.
The paper also contains some estimates of the rate of convergence in terms of the rectified modulus of continuity and the second modulus of continuity.
In a final section some illustrative examples and applications are provided.

Introduction

The representation (or the approximation) formulae for strongly continuous operator semigroups (in short, C_0-semigroups) are of interest both from a theoretical point of view and from an applied one, especially when they are involved, for instance, in the numerical analysis of the partial differential equations governed by such C_0-semigroups.

Various methods and results are known in this field, often accompanied by a through study of the rate of convergence of the given representations.

Other than functional-analytic methods, several other approaches have been developed in order to establish representations formulae for C_0-semigroups. With no claim of completeness, we quote [9, 17, 19] as general references and [12, 14, 15, 16, 24] for more specialized results. A fruitful probabilistic approach has been developed, among others, by [10, 11] and [21, 22, 23]. More recently, in [18] a unifying approach has been proposed in terms of functional calculus and Bernstein functions, while in [2] and [3] and in the references therein, approximation formulae have been established by involving iterates of positive linear operators.

Actually, many classical approximation processes acting on function spaces have suggested representation formulae for C_0-semigroups and, conversely, from them it is possible to recover approximation processes by properly specializing the underlying Banach space and the relevant C_0-semigroup (see, e.g, [9, 10, 12, 20, 22, 24]).

In the present paper we discuss some representation formulae which find their roots in some recent papers ([4, 5]) where the authors introduced and studied a new sequence of positive linear operators which act on continuous function spaces on convex compact subsets and which, among other things, generalize the classical Kantorovich operators.

The representation formulae are expressed in terms of limits of integrated means with respect to some given families of probability Borel measures and other parameters.

We discuss both the cases where these limits hold true pointwise or uniformly on compact subintervals.

The former case is developed by using purely functional-analytic methods. The latter one is tackled by involving more direct methods arising from Approximation Theory. In particular, the path we followed naturally led us to introduce a sequence of positive linear operators which extend to arbitrary intervals those studied in [4, 5, 6].

A particular case of the representation formulae developed in the present paper, can be considered as a transposition of the approximation properties of Bernstein-Schnabl operators (see [2, 3, 7]) in the semigroup theory setting.

Actually, although expressed in completely different terms, this last representation formula is, indeed, the same given in [10] but here we completely overcome the probabilistic background involved in that paper, thus by furnishing a new proof and some improvements.

However, the probabilistic background of [10] fully enters into play in Section 4 where we give some estimates of the rate of convergence in terms of the rectified modulus of continuity and the second modulus of continuity.

We end the paper with some illustrative examples and applications concerning both compact and unbounded intervals.

[a]Dipartimento di Matematica, Università degli Studi di Bari A. Moro, Campus Universitario, Via E. Orabona n. 4, 70125-Bari, Italy, e-mail: francesco.altomare@uniba.it
[b]Dipartimento di Matematica, Università degli Studi di Bari A. Moro, Campus Universitario, Via E. Orabona n. 4, 70125-Bari, Italy, e-mail: mirella.cappellettimontano@uniba.it
[c]Dipartimento di Matematica, Informatica ed Economia, Università degli Studi della Basilicata, Viale Dell' Ateneo Lucano n. 10, Campus di Macchia Romana, 85100-Potenza, Italy e-mail: vita.leonessa@unibas.it

1 Preliminaries and notation

Throughout the paper we shall consider an arbitrary subinterval I of $[0,+\infty[$; the symbol $M_+(I)$ (resp., $M_+^1(I)$) stands for the space of all Borel measures (resp., probability Borel measure) on I.

If $a \in I$, ε_a will designate the probability Borel measure on I defined by

$$\varepsilon_a(B) = \begin{cases} 1 & \text{if } a \in B; \\ 0 & \text{if } a \notin B, \end{cases}$$

for all Borel subset B of I.

For a given Banach space X, we shall denote by $\mathcal{L}(X)$ the Banach space of all bounded linear operators from X into X endowed with the usual operator norm $\|\cdot\|$.

If $T \in \mathcal{L}(X)$ and $n \geq 1$, T^n stands for the n-th iterate of T, i.e., $T^n = \underbrace{T \circ T \circ \ldots \circ T}_{n-\text{times}}$.

If $\varphi : I \to X$ is a continuous and bounded mapping and $\mu \in M_+^1(I)$, then φ is Bochner integrable since

$$\int_I \|\varphi(s)\| \, d\mu(s) < +\infty$$

(see, e.g. [19, Theorem 3.7.4] or [9, Appendix, p. 292]). We shall denote by

$$\int_I \varphi(s) \, d\mu(s) \in X$$

its Bochner integral (see, e.g., [19, Section 3.7]).

We recall that

$$\left\| \int_I \varphi(s) \, d\mu(s) \right\| \leq \int_I \|\varphi(s)\| \, d\mu(s).$$

Moreover, if $T \in \mathcal{L}(X)$, then ([19, Theorem 3.7.12])

$$T\left(\int_I \varphi(s) \, d\mu(s) \right) = \int_I T(\varphi(s)) \, d\mu(s). \tag{1}$$

In particular, if $f : I \to \mathbb{R}$ is continuous and μ-integrable and $u \in X$, then the mapping $s \in I \mapsto f(s)u$ is μ-integrable and

$$\int_I f(s)u \, d\mu(s) = \left(\int_I f(s) \, d\mu(s) \right) u. \tag{2}$$

If $(T(t))_{t\geq 0}$ is a C_0-semigroup of bounded linear operators on X (see, e.g., [17, 19] for more details on the relevant theory), we shall denote by $(A, D(A))$ its generator. We recall that

$$D(A) := \left\{ u \in X \;\middle|\; \text{there exists } \lim_{h\to 0^+} \frac{T(h)u - u}{h} \in X \right\}$$

and

$$Au := \lim_{h\to 0^+} \frac{T(h)u - u}{h} \tag{3}$$

for every $u \in D(A)$. Moreover, $D(A)$ is dense in X.

A semigroup is said to be bounded if there exists $M \geq 0$ such that, for every $t \geq 0$,

$$\|T(t)\| \leq M.$$

We also recall that, if $(T(t))_{t\geq 0}$ is a C_0-semigroup with generator $(A, D(A))$, then, for every $u \in D(A)$, the function

$$\zeta_u : t \geq 0 \mapsto T(t)u \in X$$

is differentiable and, for every $t \geq 0$, $\zeta_u'(t) = T(t)Au$. In particular, if $(T(t))_{t\geq 0}$ is bounded by some constant $M \geq 0$, then

$$\|\zeta_u'(t)\| = \|T(t)Au\| \leq M\|Au\|.$$

In such a case, taking the mean value theorem into account (see, e.g., [13, Theorem 8.5.1]) we have that ζ_u is Lipschitz-continuous with Lipschitz constant $M\|Au\|$. More precisely,

$$\|T(s)u - T(t)u\| \leq M\|Au\||s-t| \tag{4}$$

for every $s, t \geq 0$.

2 Integrated means of operator semigroups

In the present section we shall introduce and study some special sequences of integrated means of a given C_0-semigroup which, as a matter of fact, guarantee a representation/approximation of the C_0-semigroup itself.

From now on we shall fix a Banach space X and a C_0-semigroup of operators $(T(t))_{t\geq 0}$ on X with generator $(A, D(A))$.

In order to make the exposition simpler, we shall assume that the semigroup $(T(t))_{t\geq 0}$ is bounded by some constant $M \geq 0$ (see, however, Remark 4 for the general case).

Consider a parameter $a \geq 0$ and two families of measures $(\mu_t)_{t\in I}$ and $(\mu_n)_{n\geq 1}$ in $M_+^1(I)$ and assume that for every $t \in I$ and $n \geq 1$ the function e_1 defined by $e_1(s) := s$ $(s \in I)$ is both μ_t-integrable and μ_n-integrable. Furthermore, assume that there exists $C \geq 0$ such that

$$\int_I s\, d\mu_n(s) \leq C \qquad \text{for every } n \geq 1 \tag{5}$$

and that

$$\int_I s\, d\mu_t(s) = t \quad \text{for every } t \in I. \tag{6}$$

Note that, if $0 \in I$, then (6) implies that $\mu_0 = \varepsilon_0$.

We observe that, if $\mu \in M_+^1(I)$, $\alpha \geq 0$ and $u \in X$, then the function

$$s \in I \mapsto T(\alpha s)u \in X$$

is continuous and bounded and hence μ-integrable. We shall make use of the symbol $\int_I T(\alpha s)\, d\mu(s)$ to denote the linear operator

$$u \in X \mapsto \int_I T(\alpha s)u\, d\mu(s)$$

which is bounded because

$$\left\| \int_I T(\alpha s)u\, d\mu(s) \right\| \leq M\|u\|.$$

Note that, if $\mu, \nu \in M_+^1(I)$, $\alpha, \beta \geq 0$ and $u \in X$, then, taking also Fubini's theorem into account (see, e.g., [19, Theorem 3.7.13]), we get

$$\begin{aligned}\int_I d\mu(t) \int_I T(\alpha t + \beta s)u\, d\nu(s) &= \int_I T(\alpha t)\left(\int_I T(\beta s)u\, d\nu(s)\right) d\mu(t) \\ = \int_I T(\beta s)\left(\int_I T(\alpha t)u\, d\mu(t)\right) d\nu(s) &= \int_I d\nu(s) \int_I T(\alpha t + \beta s)u\, d\mu(t).\end{aligned}$$

By induction similar identities can be stated for finitely many probability Borel measures on I.

Accordingly, for every $n \geq 1$ and $t \in I$, we can consider the bounded linear operator $K_n(t) : X \to X$ defined by

$$\begin{aligned} K_n(t)u &:= \left[\int_I T\left(\frac{s}{n+a}\right) d\mu_t(s)\right]^n \circ \left[\int_I T\left(\frac{as}{n+a}\right) d\mu_n(s)\right](u) \\ &= \int_I d\mu_t(s_1) \ldots \int_I d\mu_t(s_n) \int_I T\left(\frac{s_1 + \ldots + s_n + as_{n+1}}{n+a}\right) u\, d\mu_n(s_{n+1}) \end{aligned} \tag{7}$$

for every $u \in X$.

In the special case $a = 0$ the operators $K_n(t)$ will be denoted by $B_n(t)$, i.e.,

$$\begin{aligned} B_n(t)u &= \left[\int_I T\left(\frac{s}{n}\right) d\mu_t(s)\right]^n (u) \\ &= \int_I d\mu_t(s_1) \ldots \int_I d\mu_t(s_{n-1}) \int_I T\left(\frac{s_1 + \ldots + s_n}{n}\right) u\, d\mu_t(s_n). \end{aligned} \tag{8}$$

The idea of introducing the operators $K_n(t)$ was suggested by some recent papers ([4, 5]) where the authors introduced and studied a similar (in spirit) sequence of positive linear operators on continuous function spaces which, among other things, generalize the classical Kantorovich operators. A similar parallelism connects the operator $B_n(t)$ with the sequence of the so-called Bernstein-Schnabl operators which have been extensively studied in the last three decades (see, e.g., [2, 3, 4] and the reference therein).

Theorem 2.1. *Under the assumptions that $(T(t))_{t\geq 0}$ is bounded and that (5)-(6) hold true, for every $u \in X$ and $t \in I$,*

$$T(t)u = \lim_{n\to\infty} K_n(t)u. \tag{9}$$

In particular,

$$T(t)u = \lim_{n\to\infty} B_n(t)u. \tag{10}$$

Proof. If $0 \in I$, the statement is obvious for $t = 0$ since $\mu_0 = \varepsilon_0$. Fix $t \in I, t > 0$. For any $n \geq 1$, consider the bounded linear operators $Q_n(t) : X \to X$ and $R_n : X \to X$ defined, for every $u \in X$, by

$$Q_n(t)u := \int_I T\left(\frac{s}{n+a}\right) u \, d\mu_t(s) \tag{1}$$

and

$$R_n(u) := \int_I T\left(\frac{as}{n+a}\right) u \, d\mu_n(s). \tag{2}$$

Thus, for every $0 \leq p \leq n$, we have that

$$\|Q_n^p(t)\| \leq M. \tag{3}$$

Preliminarily, we proceed to show (9) in the special case when $u \in D(A)$.

In such a case we need first to show that

$$\lim_{n\to\infty} R_n(u) = u. \tag{4}$$

Actually, taking (4) into account, for any $n \geq 1$ and $s \in I$,

$$\left\| T\left(\frac{as}{n+a}\right) u - u \right\| \leq M\|Au\| \frac{as}{n+a}.$$

Thanks to (5), for all $n \geq 1$,

$$\|R_n(u) - u\| \leq \int_I \left\| T\left(\frac{as}{n+a}\right) u - u \right\| d\mu_n(s) \leq \frac{M\|Au\|a}{n+a} C$$

and this completes the proof of (4).

Proceeding further, we point out that

$$K_n(t) = Q_n^n(t) \circ R_n = R_n \circ Q_n^n(t)$$

and hence we can write

$$K_n(t)u - T(t)u = Q_n^n(t)u - C_n^n(t)u + Q_n^n(t)(R_n(u) - u), \tag{5}$$

where $C_n(t) := T(t/n)$ $(t \in I, n \geq 1)$.

Therefore, on account of (4) and (3), it is enough to show that

$$\lim_{n\to\infty} (Q_n^n(t)u - C_n^n(t)u) = 0.$$

Since $Q_n(t)$ and $C_n(t)$ commute, we get

$$\begin{aligned}\|Q_n^n(t)u - C_n^n(t)u\| &= \left\| \sum_{p=0}^{n-1} Q_n^p(t) C_n^{n-1-p}(t) (Q_n(t)u - C_n(t)u) \right\| \\ &\leq nM\|Q_n(t)u - C_n(t)u\|.\end{aligned}$$

On the other hand

$$n(Q_n(t)u - C_n(t)u) = \int_I n\left(T\left(\frac{s}{n+a}\right)u - T\left(\frac{t}{n}\right)u\right) d\mu_t(s). \tag{6}$$

If $s, t \in I$, $s > 0$, we have

$$\begin{aligned}&n\left(T\left(\frac{s}{n+a}\right)u - T\left(\frac{t}{n}\right)u\right) \\ &= n\left(T\left(\frac{s}{n+a}\right)u - u\right) - n\left(T\left(\frac{t}{n}\right)u - u\right) \\ &= \frac{n}{n+a} s\left(\frac{T(s/(n+a))u - u}{s/(n+a)}\right) - t\left(\frac{T(t/n)u - u}{t/n}\right)\end{aligned}$$

and hence

$$\lim_{n\to\infty} n\left(T\left(\frac{s}{n+a}\right)u - T\left(\frac{t}{n}\right)u\right) = (s-t)Au. \tag{7}$$

Obviously, formula (7) holds true for $s = 0$ as well, in the case $0 \in I$.

Moreover, for $s, t \in I$ and $u \in D(A)$, taking (4) into account, we have that

$$\begin{aligned}n\left\| T\left(\frac{s}{n+a}\right)u - T\left(\frac{t}{n}\right)u \right\| &\leq nM\|Au\| \left| \frac{s}{n+a} - \frac{t}{n} \right| \\ &\leq M\|Au\|(s+t).\end{aligned}$$

Taking (7), (6) and the Lebesgue dominated convergence theorem into account, we get that

$$\lim_{n\to\infty} n(Q_n(t)u - C_n(t)u) = \int_I (s-t)Au \, d\mu_t(s) = 0. \tag{8}$$

Accordingly, from (4), (5) and (8), it follows that (9) holds true if $u \in D(A)$.

Fix now $u \in X$ and $\varepsilon > 0$. Then, there exists $v \in D(A)$ such that $\|u - v\| \le \varepsilon$. Moreover, there exists $\nu \in \mathbb{N}$ such that, if $n \ge \nu$,

$$\|K_n(t)(v) - T(t)v\| \le \varepsilon.$$

Therefore, recalling that $\|K_n(t)\| \le M$ and $\|T(t)\| \le M$ for any $n \ge 1$, we obtain

$$\begin{aligned}\|K_n(t)u - T(t)u\| &\le \|K_n(t)u - K_n(t)v\| + \|K_n(t)v - T(t)v\| \\ &+ \|T(t)v - T(t)u\| \le (2M+1)\varepsilon\end{aligned}$$

and hence the result follows. □

3 On the uniform representation on compact subintervals

In the present section we are aiming for finding some additional hypotheses under which the representation formulae given by Theorem 2.1 hold true uniformly on compact subintervals of I.

In order to achieve our result, we follow a different approach from the one of the proof of Theorem 2.1. We introduce, indeed, a sequence of positive linear operators acting on spaces of real-valued continuous functions with at most quadratic growth, whose analytic expressions are formally the same as the ones of the operators $K_n(t)$, $n \ge 1$. By means of them and a suitable device, we obtain the result.

Let I be a real interval (not necessarily contained in $[0, +\infty[$) and denote by $E_2(I)$ the linear subspace of all continuous functions $f \in C(I, \mathbb{R})$ satisfying $|f(s)| \le M(1 + s^2)$ $(s \in I)$ for some $M \ge 0$.

Clearly, $E_2(I)$ contains the functions $\mathbf{1}$, $e_1(s) := s$, $e_2(s) := s^2$ $(s \in I)$. Moreover, $E_2(I)$ also contains the subspace $C_b(I, \mathbb{R})$ of all real-valued bounded continuous functions on I.

Consider two families of measures $(\mu_t)_{t \in I}$ and $(\mu_n)_{n \ge 1}$ in $M_1^+(I)$, together with a positive constant a. Assume that

$$e_2 \in \mathcal{L}^1(\mu_t) \cap \mathcal{L}^1(\mu_n) \quad \text{for every } t \in I \text{ and } n \ge 1. \tag{11}$$

From the inequality $|e_1| \le 1 + e_2$ it also follows that $e_1 \in \mathcal{L}^1(\mu_t) \cap \mathcal{L}^1(\mu_n)$ $(t \in I, n \ge 1)$.

For every $n \ge 1$, $f \in E_2(I)$, and $t \in I$ set

$$C_n(f)(t) := \int_I d\mu_t(s_1) \ldots \int_I d\mu_t(s_n) \int_I f\left(\frac{s_1 + \ldots + s_n + a s_{n+1}}{n+a}\right) d\mu_n(s_{n+1}). \tag{12}$$

The multiple integral in (12) is convergent because, considering a constant M such that $|f| \le M(1 + e_2)$, for every $s_1, \ldots, s_{n+1} \in I$, we have

$$\begin{aligned}\left|f\left(\frac{s_1 + \ldots + s_n + a s_{n+1}}{n+a}\right)\right| &\le M\left(1 + \left(\frac{s_1 + \ldots + s_n + a s_{n+1}}{n+a}\right)^2\right) \\ &\le M\left(1 + \frac{s_1^2 + \ldots + s_n^2 + a s_{n+1}^2}{n+a}\right)\end{aligned}$$

and hence

$$\begin{aligned}&\int_I d\mu_t(s_1) \ldots \int_I d\mu_t(s_n) \int_I \left|f\left(\frac{s_1 + \ldots + s_n + a s_{n+1}}{n+a}\right)\right| d\mu_n(s_{n+1}) \\ &\le M\left(1 + \frac{1}{n+a}\left(n \int_I s^2 \, d\mu_t(s) + a \int_I s^2 \, d\mu_n(s)\right)\right).\end{aligned}$$

In the case $a = 0$, the operators C_n $(n \ge 1)$ turn into the operators

$$B_n(f)(t) = \int_I d\mu_t(s_1) \ldots \int_I f\left(\frac{s_1 + \ldots + s_n}{n}\right) d\mu_t(s_n) \tag{13}$$

$(f \in E_2(I),\ t \in I)$.

The operators B_n extend to the framework of arbitrary intervals the sequence of Bernstein-Schnabl operators and they have been first studied in [6].

The operators C_n extend to the setting of arbitrary intervals the generalized Kantorovich operators which have been introduced and studied in [4] and [5] in the context of compact real intervals (actually, in the more general context of convex compact subsets of locally convex spaces).

In the case $I = [0, 1]$, under a special choice of the measure μ_t $(0 \le t \le 1)$, the operators B_n and C_n turn into the classical Bernstein operators and the Kantorovich operators, respectively (see the final Section 5).

From now on, together with (6), we shall further assume that

$$\sup_{t \in J} \int_I s^2 \, d\mu_t(s) < +\infty \text{ for every compact subinterval } J \text{ of } I \tag{14}$$

and

$$\sup_{n\geq 1}\int_I s^2\,d\mu_n(s)<+\infty. \tag{15}$$

As pointed out before, from (15) it also follows that

$$\sup_{n\geq 1}\int_I |s|\,d\mu_n(s)<+\infty.$$

In the present framework a useful role is played by the family of functions $(\psi_t)_{t\in I}$ defined by

$$\psi_t(s)=|s-t| \qquad (s,t\in I). \tag{16}$$

Clearly, $\psi_t^2\in E_2(I)$ for every $t\in I$.

Proposition 3.1. *Assume that (6) and (11) hold true. Under the further assumption (14) if $a=0$ and (14)-(15) if $a>0$, the following properties hold true:*

(1) $\lim_{n\to\infty} C_n(\psi_t^2)(t)=0$ uniformly on compact subintervals of I.

(2) If $f\in C_b(I,\mathbb{R})$, then $\lim_{n\to\infty} C_n(f)=f$ uniformly on compact subintervals of I.

Proof. As regards Part (1), direct calculations show that, for every $n\geq 1$ and $t\in I$,

$$C_n(\mathbf{1})=\mathbf{1},$$

$$C_n(e_1)(t)=\frac{n}{n+a}t+\frac{a}{n+a}\int_I s\,d\mu_n(s),$$

and

$$\begin{aligned}C_n(e_2)(t)&=\frac{a^2}{(n+a)^2}\int_I s^2\,d\mu_n(s)+\frac{2nat}{(n+a)^2}\int_I s\,d\mu_n(s)\\&+\frac{n}{(n+a)^2}\int_I s^2\,d\mu_t(s)+\frac{n(n-1)}{(n+a)^2}t^2.\end{aligned}$$

Therefore,

$$\begin{aligned}C_n(\psi_t^2)(t)&=\frac{a^2}{(n+a)^2}\int_I s^2\,d\mu_n(s)-\frac{2a^2t}{(n+a)^2}\int_I s\,d\mu_n(s)\\&+\frac{n}{(n+a)^2}\int_I s^2\,d\mu_t(s)-\frac{n-a^2}{(n+a)^2}t^2\end{aligned}$$

and hence the result follows.

Part (2) is an immediate consequence of Part (1) and Theorem 3.5 of [1]. □

Remark 1.

1. As the proof above shows, in the case $a>0$, Part (1) of Proposition 3.1 (and hence Part (2) as well) continue to hold true by replacing condition (15) with the weaker one

$$\lim_{n\to\infty}\frac{1}{n^2}\int_I s^2\,d\mu_n(s)=0,$$

which in turn also implies that

$$\lim_{n\to\infty}\frac{1}{n^2}\int_I |s|\,d\mu_n(s)=0.$$

2. Actually, solely under the assumptions (5) and (6) and for $I=[0,+\infty[$, we get that for every $f\in UC_b([0,+\infty[)$ and $t\geq 0$

$$\lim_{n\to\infty} C_n(f)(t)=f(t)$$

and, in particular,

$$\lim_{n\to\infty} B_n(f)(t)=f(t).$$

These convergence formulas can be easily proved as follows. Consider the Banach space $X:=UC_b([0,+\infty[)$ of all uniformly continuous and bounded functions on $[0,+\infty[$ endowed with the sup-norm and denote by $(T(t))_{t\geq 0}$ the translation semigroup defined on it, i.e., for every $t\geq 0$, $f\in UC_b([0,+\infty[)$ and $x\geq 0$,

$$T(t)f(x)=f(x+t).$$

If $(\mu_t)_{t\geq 0}$ and $(\mu_n)_{n\geq 1}$ are families of probability Borel measures on $[0,+\infty[$ satisfying (5)-(6), from Theorem 2.1 it follows that, for any $f\in UC_b([0,+\infty[$ and $t\in[0,+\infty[$,

$$f(x+t)=\lim_{n\to\infty}\int_0^{+\infty}d\mu_t(s_1)\ldots\int_0^{+\infty}f\left(x+\frac{s_1+\ldots+s_n}{n}\right)d\mu_t(s_n) \tag{17}$$

and

$$\begin{aligned}&f(x+t)\\&=\lim_{n\to\infty}\int_0^{+\infty}d\mu_t(s_1)\ldots\int_0^{+\infty}d\mu_t(s_n)\int_0^{+\infty}f\left(x+\frac{s_1+\ldots+s_n+as_{n+1}}{n+a}\right)d\mu_n(s_{n+1})\end{aligned} \tag{18}$$

uniformly w.r.t. $x\geq 0$.

In particular, for $x=0$, we obtain

$$f(t)=\lim_{n\to\infty}\int_0^{+\infty}d\mu_t(s_1)\ldots\int_0^{+\infty}f\left(\frac{s_1+\ldots+s_n}{n}\right)d\mu_t(s_n)=\lim_{n\to\infty}B_n(f)(t) \tag{19}$$

and

$$\begin{aligned}f(t)&=\lim_{n\to\infty}\int_0^{+\infty}d\mu_t(s_1)\ldots\int_0^{+\infty}d\mu_t(s_n)\int_0^{+\infty}f\left(\frac{s_1+\ldots+s_n+as_{n+1}}{n+a}\right)d\mu_n(s_{n+1})\\&=\lim_{n\to\infty}C_n(f)(t).\end{aligned} \tag{20}$$

Thus, in addition to the operators B_n which have been already studied in [6], the operators C_n, $n\geq 1$, represent another approximation process and they seem to have an independent interest on their own. Their study will be throughly deepened in a forthcoming paper.

Proposition 3.1 and the next result play a key role to obtain a uniform representation of bounded C_0-semigroups on compact subintervals.

Lemma 3.2. *Consider a normed space $(X,\|\cdot\|)$ and a real interval I. If $F:I\to X$ is continuous and bounded, then for every $\varepsilon>0$ and for every compact subinterval J of I there exists $\delta>0$ such that*

$$\|F(s)-F(t)\|\leq\varepsilon+\frac{2\|F\|_\infty}{\delta^2}\psi_t^2(s) \tag{21}$$

for every $t\in J$ and $s\in I$, where $\|F\|_\infty=\sup_{s\in I}\|F(s)\|$.

Proof. Given $\varepsilon>0$ and a compact subinterval $J\subset I$, we first proceed to show that there exists $\delta>0$ such that

$$\|F(s)-F(t)\|\leq\varepsilon\quad\text{for every } s\in I, t\in J, |s-t|\leq\delta. \tag{1}$$

Suppose, on the contrary, that (1) is not true. Then there exist two sequences $(s_n)_{n\geq 1}$ in I and $(t_n)_{n\geq 1}$ in J such that

$$|s_n-t_n|\leq\frac{1}{n}\quad\text{and}\quad\|F(s_n)-F(t_n)\|>\varepsilon\quad\text{for every } n\geq 1. \tag{2}$$

Because of the compactness of J, there exists a subsequence $(t_{k(n)})_{n\geq 1}$ converging to some $t_0\in J$. Then necessarily $(s_{k(n)})_{n\geq 1}$ converges to t_0 as well, since, for every $n\geq 1$,

$$|s_{k(n)}-t_0|\leq|s_{k(n)}-t_{k(n)}|+|t_{k(n)}-t_0|\leq\frac{1}{n}+|t_{k(n)}-t_0|.$$

The continuity of F at t_0 implies that $F(t_{k(n)})\to F(t_0)$ and $F(s_{k(n)})\to F(t_0)$ as $n\to\infty$ which contradicts (2).

Now, it is clear that, if $t\in J$ and $s\in I$, then $\|F(s)-F(t)\|\leq\varepsilon$ if $|s-t|\leq\delta$, whereas, if $|s-t|\geq\delta$, then

$$\|F(s)-F(t)\|\leq 2\|F\|_\infty\leq\frac{2\|F\|_\infty}{\delta^2}\psi_t^2(s)$$

and so the result follows. □

We are now in the position to state and prove the result we are aiming for, which concerns the uniform representation on compact subintervals in terms of the integrated means introduced in Section 2.

Theorem 3.3. *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a Banach space X and let $I\subset[0,+\infty[$ be a real interval. Consider some sequences $(\mu_n)_{n\geq 1}$ and $(\mu_t)_{t\in I}$ of probability Borel measures on I satisfying (6) and (11), a real number $a\geq 0$, and consider the relevant operator $(K_n(t))_{n\geq 1}$ and $(B_n(t))_{n\geq 1}$ defined by (7) and (8) for every $t\in I$.*

Given $u\in X$ and a compact subinterval J of I, the following properties hold true:

1) If $a=0$, under the further assumption (14), then

$$T(t)u=\lim_{n\to\infty}B_n(t)u\quad\textit{uniformly w.r.t. } t\in J.$$

2) If $a > 0$, under the further assumptions (14)-(15), then

$$T(t)u = \lim_{n\to\infty} K_n(t)u \quad \text{uniformly w.r.t. } t \in J.$$

Proof. Given $u \in X$, the function $F : I \to X$ defined by $F(t) := T(t)u$ $(t \in I)$ is continuous and bounded and $\|F\|_\infty \le M\|u\|$, where M is a bound for the semigroup $(T(t))_{t\ge0}$.

By Lemma 3.2, given $\varepsilon > 0$, there exists $\delta > 0$ such that (21) is satisfied.

Therefore, for every $t \in J$ and $n \ge 1$ we get (in both cases $a = 0$ and $a > 0$)

$$\begin{aligned}
&\|K_n(t)u - T(t)u\| \\
&\le \int_I d\mu_t(s_1)\dots\int_I d\mu_t(s_n)\int_I \left\| F\left(\frac{s_1+\dots s_n+as_{n+1}}{n+a}\right) - F(t)\right\| d\mu_n(s_{n+1}) \\
&\le \varepsilon + \frac{2M\|u\|}{\delta^2}\int_I d\mu_t(s_1)\dots\int_I d\mu_t(s_n)\int_I \psi_t^2\left(\frac{s_1+\dots s_n+as_{n+1}}{n+a}\right) d\mu_n(s_{n+1}) \\
&= \varepsilon + \frac{2M\|u\|}{\delta^2} C_n(\psi_t^2)(t).
\end{aligned}$$

Now the conclusion can be easily achieved by using Part (1) of Proposition 3.1. We omit the details for the sake of brevity. □

4 Estimates of the rate of convergence

In this section we present some estimates of the rate of convergence for the representation formulae given in Theorem 2.1 (resp., Theorem 3.3) in the special case $e_2 \in \bigcap_{t\in I} \mathcal{L}^1(\mu_t)$.

To this end we shall make use of a probabilistic representation of the operators $B_n(t)$ in terms of the expected values of the arithmetic means of suitable random variables together with an estimate given in [10] in such setting.

The estimates will be given in terms of the rectified modulus of continuity and the modulus of continuity of order two defined, respectively, by

$$\omega_1(u,\delta) := \sup_{\substack{s,t\ge0\\|s-t|\le\delta}} \|T(t)u - T(s)u\| \tag{22}$$

and

$$\omega_2(u,\delta) := \sup_{0\le t\le\delta} \|(T(t)-I_X)^2u\| \tag{23}$$

$(u \in X,\ \delta \ge 0)$ (see [9, p. 19], [10, formula (2.1)]) where I_X denotes the identity operator on X. The next lemma will simplify the proof of the main result of this section.

Lemma 4.1. *Under the same assumptions of Theorem 2.1, consider the bounded linear operators $Q_n(t)$ and R_n defined within the proof of Theorem 2.1 (see formulae (1) and (2), respectively) and $B_n(t)$ (see (8)). Then, for every $n \ge 1$, $t \in I$, $\delta \ge 0$ and $u \in X$, the following estimates hold true:*

(i) $\|Q_n(t)^n u - B_n(t)u\| \le (1+t)\,\omega_1\left(u, \frac{a}{n+a}\right)$;

(ii) $\omega_1(R_n(u),\delta) \le \omega_1(u,\delta)$;

(iii) $\omega_2(R_n(u),\delta) \le M\omega_2(u,\delta)$;

(iv) $\|R_n(u) - u\| \le \left(1 + \int_I s\,d\mu_n(s)\right)\omega_1\left(u, \frac{a}{n+a}\right)$.

Proof. (i) If $a = 0$ we have nothing to show as, in such a case, $Q_n(t)^n = B_n(t)$. If $a > 0$, we have that

$$\begin{aligned}
&\|Q_n(t)^n u - B_n(t)u\| \le \\
&\le \int_I d\mu_t(s_1)\dots\int_I \left\| T\left(\frac{s_1+\dots+s_n}{n+a}\right)u - T\left(\frac{s_1+\dots+s_n}{n}\right)u\right\| d\mu_t(s_n) \\
&\le \omega_1\left(u, \frac{ta}{n+a}\right) \le (1+t)\,\omega_1\left(u, \frac{a}{n+a}\right).
\end{aligned}$$

(ii) For $\xi, \eta \ge 0$, $|\xi - \eta| \le \delta$, we have

$$\begin{aligned}
&\|T(\xi)R_n(u) - T(\eta)R_n(u)\| \le \\
&\le \int_I \left\| T\left(\xi + \frac{as}{n+a}\right)u - T\left(\eta + \frac{as}{n+a}\right)u\right\| d\mu_n(s) \le \omega_1(u,\delta)
\end{aligned}$$

and hence the result follows.
(iii) If $0 \le \xi \le \delta$, then

$$\begin{aligned}\|(T(\xi)-I_X)^2 R_n(u)\| &= \left\| \int_I (T(\xi)-I_X)^2 \left(T\left(\frac{as}{n+a}\right) u \right) d\mu_n(s) \right\| \\ &= \left\| \int_I T\left(\frac{as}{n+a}\right) (T(\xi)-I_X)^2 u \, d\mu_n(s) \right\| \\ &\le \int_I \left\| T\left(\frac{as}{n+a}\right) \right\| \left\| (T(\xi)-I_X)^2 u \right\| d\mu_n(s) \le M \omega_2(u,\delta),\end{aligned}$$

which establishes the formula.
(iv) Estimate (iv) easily follows because

$$\|R_n(u)-u\| \le \int_I \left\| T\left(\frac{as}{n+a}\right) u - u \right\| d\mu_n(s)$$

and

$$\left\| T\left(\frac{as}{n+a}\right) u - u \right\| \le \omega_1\left(u, \frac{sa}{n+a}\right) \le (1+s)\ \omega_1\left(u, \frac{a}{n+a}\right).$$

□

Remark 2. We point out that, if $I=[0,1]$, then Statement (i) in the above lemma can be replaced by

$$\|Q_n(t)^n u - B_n(t)u\| \le \ \omega_1\left(u, \frac{a}{n+a}\right) \quad (t \in [0,1], n \ge 1).$$

In the same way, Statement (iv) turns into

$$\|R_n(u)-u\| \le \omega_1\left(u, \frac{a}{n+a}\right) \quad (n \ge 1).$$

Next we proceed to show the main result of the present section. For its proof we shall make use of the probabilistic background and the main result of [10]. For unexplained probabilistic terminology we refer, e.g., to [8].

Theorem 4.2. *Under the assumptions of Theorem 2.1, further assume that $e_2 \in \mathcal{L}^1(\mu_t)$ for every $t \in I$. Then, for every $u \in X$, $n \ge 1$ and $t \in I$, the following estimates hold true:*

$$\|B_n(t)u - T(t)u\| \le M_1 \omega_2\left(u, \sqrt{\frac{\beta_2(t)+t^2}{n}}\right) \tag{24}$$

and

$$\begin{aligned}&\|K_n(t)u - T(t)u\| \\ &\le M_1 \omega_2\left(u, \sqrt{\frac{\beta_2(t)+t^2}{n}}\right) + M\left(2+t+\int_I s\, d\mu_n(s)\right) \omega_1\left(u, \frac{a}{n+a}\right),\end{aligned} \tag{25}$$

where M_1 is a constant independent of u, n and t and $\beta_2(t) := \int_I s^2\, d\mu_t(s)$ $(t \in I)$.

Proof. At first we assume that $(T(t))_{t\ge 0}$ is contractive, i.e., $\|T(t)\| \le 1$ for every $t \ge 0$, and hence $M \le 1$. For a given $t \in I$ consider a sequence $(X_{n,t})_{n\ge 1}$ of independent identically distributed random variables from a suitable probability space $(\Omega, \mathcal{F}, P)$ into I such that all the distributions $P_{X_{n,t}}$ are equal to μ_t (see, e.g., [8, Corollary 9.5]).

Then the expected values $E(X_{n,t})$ of $X_{n,t}$ are equal to $\int_I s\, d\mu_t(s) = t$ and $E(X_{n,t}^2) = \int_I s^2\, d\mu_t(s) = \beta_2(t)$.

Setting $S_{n,t} := \frac{1}{n}\sum_{k=1}^{n} X_{k,t}$ $(n \ge 1)$, for every $u \in X$ we get

$$\begin{aligned}E\left(T\left(\frac{S_{n,t}}{n}\right)u\right) &= \int_\Omega T\left(\frac{1}{n}\sum_{k=1}^{n} X_{k,t}(\omega)\right) u\, dP(\omega) \\ &= \int_I dP_{X_{1,t}}(s_1) \ldots \int_I T\left(\frac{s_1+\ldots+s_n}{n}\right) u\, dP_{X_{n,t}}(s_n) = B_n(t)u\,.\end{aligned}$$

Therefore, formula (24) directly follows from [10, Theorem 1, formula (3.2)].

In order to show (25), introducing the operators $Q_n(t)$ and R_n as in the proof of Theorem 2.1, we get

$$\begin{aligned}K_n(t)u - T(t)u &= Q_n(t)^n(R_n(u)) - T(t)u\\ &= Q_n(t)^n(R_n(u)) - B_n(t)(R_n(u))\\ &+ B_n(t)(R_n(u)) - T(t)(R_n(u)) + T(t)(R_n(u) - u);\end{aligned}$$

hence, taking (24) and Lemma 4.1 into account, and recalling that $M \le 1$, we get

$$\begin{aligned}&\|K_n(t)u - T(t)u\| \le\\ &\le \|Q_n(t)^n(R_n(u)) - B_n(t)(R_n(u))\| + M_1\omega_2\left(R_n(u), \sqrt{\frac{\beta_2(t)+t^2}{n}}\right)\\ &+ \|R_n(u) - u\| \le (1+t)\,\omega_1\left(R_n(u), \frac{a}{n+a}\right)\\ &+ M_1\omega_2\left(u, \sqrt{\frac{\beta_2(t)+t^2}{n}}\right) + \left(1 + \int_I s\,d\mu_n(s)\right)\omega_1\left(u, \frac{a}{n+a}\right)\\ &\le M_1\omega_2\left(u, \sqrt{\frac{\beta_2(t)+t^2}{n}}\right) + \left(2 + t + \int_I s\,d\mu_n(s)\right)\omega_1\left(u, \frac{a}{n+a}\right),\end{aligned}$$

and the proof of (25) is now complete, for the contractive case.

In the general case where $(T(t))_{t\ge0}$ is bounded, it is enough to consider the new norm on X defined by

$$|||u||| := \sup_{0\le t}\|T(t)u\| \quad (u \in X).$$

The norm $|||\cdot|||$ is equivalent to $\|\cdot\|$ because, for every $u \in X$,

$$\|u\| \le |||u||| \le M\|u\|. \tag{1}$$

Moreover, each $T(t)$ is contractive with respect to $|||\cdot|||$. By applying (24) and (25) with respect to the norm $|||\cdot|||$, we get the result by taking (1) into account and by observing that the moduli of continuity ω_1 and ω_2 do not change up to the multiplicative constant M. □

Remark 3. Under the same assumptions of Theorem 4.2, if $I = [0,1]$, then the estimate (25) can be simplified as:

$$\|K_n(t)u - T(t)u\| \le M_1\omega_2\left(u, \sqrt{\frac{\beta_2(t)+t^2}{n}}\right) + 2M\omega_1\left(u, \frac{a}{n+a}\right).$$

Remark 4. If $(T(t))_{t\ge0}$ is an arbitrary C_0-semigroup on X, then there exist $M \ge 0$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \le Me^{\omega t}$ $(t \ge 0)$ (see, e.g. [17, Proposition 5.5]). Therefore, by applying Theorem 2.1 and Theorem 4.2 to the bounded C_0-semigroup $S(t) := e^{-\omega t}T(t)$ $(t \ge 0)$, it is also possible to obtain estimates for $T(t) = e^{\omega t}S(t)$ $(t \ge 0)$ in terms of the approximating operators

$$\widetilde{K}_n(t) := e^{\omega t}\left[\int_I e^{-\frac{\omega s}{n+a}}T\left(\frac{s}{n+a}\right)d\mu_t(s)\right]^n \circ \left[\int_I e^{-\frac{\omega a s}{n+a}}T\left(\frac{as}{n+a}\right)d\mu_n(s)\right]$$

and the moduli of continuity $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ defined by means of the C_0-semigroup $(S(t))_{t\ge0}$. We omit the details for the sake of simplicity.

5 Examples and applications

In this final section we apply the main results of the previous sections in order to show some new representation formulae which are obtained by specifying the measures μ_t, $t \in I$, and μ_n, $n \ge 1$. For the sake of brevity we chose few examples but, obviously, many other ones can be furnished due to the generality of our approach.

To make the examples more transparent, it is useful to give a description of the representing operators $B_n(t)$ and $K_n(t)$ in terms of another sequence of linear operators acting on spaces of vector valued continuous functions.

More precisely, by referring to the same notation of Section 2, for every continuous and bounded function $F : I \to X$, $n \ge 1$, and $t \in I$, set

$$B_n^*(F)(t) := \int_I d\mu_t(s_1)\ldots\int_I F\left(\frac{s_1+\ldots+s_n}{n}\right)d\mu_t(s_n).$$

Then, for a given $u \in X$, we get

$$B_n(t)u = B_n^*(T(\cdot)u)(t) \tag{26}$$

and

$$C_n(t)u = B_n^*(I_{n,u})(t), \tag{27}$$

where

$$I_{n,u}(t) = \int_I T\left(\frac{nt+as}{n+a}\right)u\,d\mu_n(s) \qquad (t \in I). \tag{28}$$

The usefulness of formulae (26) and (27) relies on the fact that the operators B_n^* are the transposition of Bernstein-Schnabl operators ([2, 3, 6, 7]) from the setting of real-valued continuous functions to the vector valued one. This new setting does not affect their formal analytic expressions which, therefore, can be used to describe both the operators $B_n(t)$ and $K_n(t)$.

5.1 Compact intervals

Let's start our analysis by assuming that I is a compact interval, for example $I = [0,1]$.

Consider the family of probability Borel measures $(\mu_t)_{0\le t\le 1}$ defined by

$$\mu_t := (1-t)\varepsilon_0 + t\varepsilon_1 \quad (0 \le t \le 1). \tag{29}$$

Then for every continuous function $F : [0,1] \to X$, $B_n^*(F)$ turns into the n-th Bernstein operator attached to F, i.e.,

$$B_n^*(F)(t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} F\left(\frac{k}{n}\right),$$

(see, e.g., [3, Section 3.1.1] or [2, pp. 295–298]) and hence, for every $t \in [0,1]$, $n \ge 1$ and $u \in X$,

$$B_n(t)u = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} T\left(\frac{k}{n}\right) u \tag{30}$$

and

$$K_n(t)u = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} \int_0^1 T\left(\frac{k+as}{n+a}\right) u \, d\mu_n(s). \tag{31}$$

In particular, if all the measures μ_n are equal to the Borel-Lebesgue measure on $[0,1]$, we obtain, for $a > 0$,

$$K_n(t)u = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} \frac{n+a}{a} \int_{\frac{k}{n+a}}^{\frac{k+a}{n+a}} T(\xi) u \, d\xi.$$

which, for $a = 1$, are the formal analog of Kantorovich operators (see, e.g., [2, pp 333-335] or [4, Example 3.1]).

Another special case of (31) can be obtained considering $\mu_n := \varepsilon_{b_n/a}$ $(n \ge 1)$, where $(b_n)_{n\ge1}$ is an arbitrary sequence in $]0,a]$. In such a case, we get

$$K_n(t)u = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} T\left(\frac{k+b_n}{n+a}\right) u \tag{32}$$

$(n \ge 1, 0 \le t \le 1, u \in X)$.

We remark that (5) and (6) are satisfied because $\beta_2(t) = t$ $(0 \le t \le 1)$. Therefore Theorems 2.1, 3.3 and 4.2 apply. Within this same framework Part 1) of Theorem 3.3 has been first obtained in [20].

Estimates (24) and (25) become, respectively,

$$\|B_n(t)u - T(t)u\| \le M_1 \omega_2\left(u, \sqrt{\frac{t+t^2}{n}}\right) \tag{33}$$

and

$$\|K_n(t)u - T(t)u\| \le M_1 \omega_2\left(u, \sqrt{\frac{t+t^2}{n}}\right) + 2M\omega_1\left(u, \frac{a}{n+a}\right) \tag{34}$$

(see also Remark 3). Estimates (33) has been first obtained in [10, p. 263, Corollary 1].

5.2 Unbounded intervals

1. Assume $I = [0,+\infty[$ and consider the family of probability Borel measures $(\mu_t)_{t\ge0}$ defined by setting

$$\mu_t := e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \varepsilon_k \qquad (t \ge 0). \tag{35}$$

Then, for each continuous and bounded function $F : [0,+\infty[\to X$,

$$B_n^*(F)(t) = e^{-nt} \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} F\left(\frac{k}{n}\right) \qquad (t \ge 0) \tag{36}$$

(Szász-Mirakjan vector valued operators) (see, e.g., [6, Examples 3.1,2]) and hence, for every $t \ge 0$, $n \ge 1$ and $u \in X$,

$$B_n(t)u = e^{-nt} \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} T\left(\frac{k}{n}\right) u \tag{37}$$

and

$$K_n(t)u = e^{-nt} \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} \int_0^{+\infty} T\left(\frac{k+as}{n+a}\right) u \, d\mu_n(s). \tag{38}$$

In particular, if μ_n is the density measure, having density the characteristic function $\mathbf{1}_{[0,1]}$ of the interval $[0,1]$, with respect to the Borel-Lebesgue measure λ_1 on $[0,+\infty[$, then

$$K_n(t)u = e^{-nt}\sum_{k=0}^{\infty}\frac{(nt)^k}{k!}\frac{n+a}{a}\int_{\frac{k}{n+a}}^{\frac{k+a}{n+a}} T(\xi)u\,d\xi. \tag{39}$$

The family $(\mu_t)_{t\geq 0}$ satisfies assumptions (6) and (14)-(15). In particular, $\beta_2(t) = t^2 + t$ for all $t \geq 0$. Hence, for every $u \in X$,

$$T(t) = \lim_{n\to\infty} B_n(t)u = \lim_{n\to\infty} e^{-nt}\sum_{k=0}^{\infty}\frac{(nt)^k}{k!}T\left(\frac{k}{n}\right)u \tag{40}$$

uniformly on compact subintervals of $[0,+\infty]$ and (see (24))

$$\|B_n(t)u - T(t)u\| \leq M_1\omega_2\left(u, \sqrt{\frac{2t^2+t}{n}}\right). \tag{41}$$

Similarly, for every $u \in X$,

$$T(t) = \lim_{n\to\infty} K_n(t)u = \lim_{n\to\infty} e^{-nt}\sum_{k=0}^{\infty}\frac{(nt)^k}{k!}\frac{n+a}{a}\int_{\frac{k}{n+a}}^{\frac{k+a}{n+a}} T(\xi)u\,d\xi \tag{42}$$

uniformly on compact subintervals of $[0,+\infty]$ and (see (25))

$$\|K_n(t)u - T(t)u\| \leq M_1\omega_2\left(u, \sqrt{\frac{2t^2+t}{n}}\right) + M\left(\frac{5}{2}+t\right)\omega_1\left(u, \frac{a}{n+a}\right). \tag{43}$$

Formula (40) is the well-known first exponential formula and estimate (41) has been first established in [10, Theorem 3].

2. For every $t \geq 0$ set

$$\mu_t := \begin{cases} \varphi(t,\cdot)\lambda_1 & \text{if } t > 0; \\ \varepsilon_0 & \text{if } t = 0, \end{cases} \tag{44}$$

where λ_1 denotes the Borel-Lebesgue measure on $[0,+\infty[$ and the function $\varphi(t,\cdot)$ is defined on $[0,+\infty[$ by

$$\varphi(t,s) := \begin{cases} \frac{e^{-s/t}}{t} & \text{if } s > 0; \\ \varepsilon_0 & \text{if } s = 0, \end{cases} \tag{45}$$

for every $t \geq 0$.

Then for every continuous and bounded function $F : [0,+\infty[\to X$ and for every $t \geq 0$,

$$B_n^*(F)(t) := \begin{cases} \frac{n^n}{t^n(n-1)!}\int_0^{+\infty} x^{n-1}e^{-\frac{nx}{t}}F(x)\,dx & \text{if } t > 0; \\ F(0) & \text{if } t = 0, \end{cases} \tag{46}$$

(Post-Widder vector valued operators) (see, e.g., [6, Examples 3.1,4]).

Therefore, for every $u \in X$, $n \geq 1$ and $t \geq 0$,

$$B_n(t)u := \begin{cases} \frac{n^n}{t^n(n-1)!}\int_0^{+\infty} x^{n-1}e^{-\frac{nx}{t}}T(x)u\,dx & \text{if } t > 0; \\ u & \text{if } t = 0, \end{cases} \tag{47}$$

and

$$K_n(t)u := \begin{cases} \frac{n^n}{t^n(n-1)!}\int_0^{+\infty} dx\int_0^{+\infty} x^{n-1}e^{-\frac{nx}{t}}T\left(\frac{nx+as}{n+a}\right)u\,d\mu_n(s) & \text{if } t > 0; \\ \int_0^{+\infty} T\left(\frac{as}{n+a}\right)u\,d\mu_n(s) & \text{if } t = 0. \end{cases} \tag{48}$$

In this case, (6) and (14) are satisfied since $\beta_2(t) = 2t^2$ $(t \geq 0)$.

If, for instance, we choose $\mu_n = \varepsilon_{b_n}$, where $b_n \geq 0$ and $\sup_{n\geq 1} b_n < +\infty$ (resp., $\sup_{n\geq 1} b_n^2 < +\infty$), then (5) (resp., (15)) is satisfied.

In such a case we have

$$K_n(t)u := \begin{cases} \frac{n^n}{t^n(n-1)!}\int_0^{+\infty} x^{n-1}e^{-\frac{nx}{t}}T\left(\frac{nx+ab_n}{n+a}\right)u\,dx & \text{if } t > 0; \\ T\left(\frac{ab_n}{n+a}\right)u & \text{if } t = 0 \end{cases} \tag{49}$$

and

$$\|B_n(t)u - T(t)u\| \leq M_1\omega_2\left(u, t\sqrt{\frac{3}{n}}\right), \tag{50}$$

as well as

$$\|K_n(t)u - T(t)u\| \leq M_1\omega_2\left(u, t\sqrt{\frac{3}{n}}\right) + M(2+t+b_n)\omega_1\left(u, \frac{a}{n+a}\right). \tag{51}$$

Estimate (50) has been first obtained in [10, Theorem 4].

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