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Lagrange polynomials of lower sets

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Abstract

A lower set of nodes is a subset of a grid that can be indexed by a lower set of indices. In order to apply the Lagrange interpolation formula, it is convenient to express the Lagrange fundamental polynomials as sums of few terms. We present such a formula for the Lagrange interpolation formula in two variables. In the general multidimensional case, we express the Lagrange fundamental polynomials in d variables in terms of Lagrange fundamental polynomials in $d-1$ variables. Applications to the problem of computing Lebesgue constants of lower sets are included.

Keywords: grid interpolation, lower sets, Lagrange formula.

1 Introduction

Interpolation problems in several variables on subspaces of polynomials are much harder to solve than univariate ones. In contrast to the univariate case, the question of the existence and uniqueness is not automatic. It is therefore important to pay attention to the distribution of nodes if we want to interpolate with a given subspace of polynomials or to select an appropriate interpolation space for a given set of nodes.

The Lagrange formula is very useful because it expresses the interpolant in terms of the data. It might lead to remarkable representations of polynomials. According to a comment at the end of Chapter 1 of [1], the univariate Lagrange formula based on the Chebyshev sites is one of the best conditioned polynomial representations available. Explicit formulae for the Lagrange polynomials also provide information on the Lebesgue constant, which can be used to describe the stability properties of the interpolation problem.

Some problems have a remarkable structure that can be exploited in order to reduce the interpolation problem to interpolation problems in lower dimensions. The tensor product construction selects a polynomial in the space $P_{n_1} \otimes \cdots \otimes P_{n_d}$ interpolating a function at a grid of points $X_1 \times \cdots \times X_d$ by reducing the multivariate problem to univariate interpolation problems. Direct generalizations of the univariate Lagrange and Newton formulae and the Aitken-Neville recurrences can be described for these problems. In these kinds of constructions, the Lagrange fundamental polynomials are products of linear factors.

Other examples of sets of nodes with simple Lagrange formulae in P_n^d, the space of polynomials in d variables of total degree less than or equal to n, are GC sets. A GC set $X \subset \mathbb{R}^d$ is a set with $\binom{n+d}{d}$ nodes such that, for each $x \in X$, there exist n hyperplanes $H_1^x, \dots, H_n^x$ such that $(H_1^x \cup \cdots \cup H_n^x) \cap X = X \setminus \{x\}$. For these sets of points the Lagrange fundamental polynomials can be expressed as a product of linear factors [3]. Principal lattices and their generalizations [2] are particular instances of GC sets and hence their Lagrange fundamental polynomials are also products of linear factors.

Multivariate Lagrange formulae have some drawbacks for general sets of nodes. Since $\dim P_n^d = \binom{n+d}{d}$, the expansion of each fundamental polynomial with respect to a suitable basis will have $\binom{n+d}{d}$ terms and the Lagrange interpolation formula combining the values at the nodes with the fundamental polynomials, will have $\binom{n+d}{d}^2$ terms. The roundoff error of the evaluation of a formula with so many terms can be large. The computation time will grow fast with the degree and the dimension. This huge number of terms hinders the practical use of Lagrange formulae in problems where the degree is not very low and dimension $d \geq 2$. Furthermore, the fundamental polynomials are usually obtained as the solution of an ill-conditioned problem and its construction might lack reliability. For this reason it is important to identify particular sets of nodes where the Lagrange fundamental polynomials are particularly simple and can be explicitly expressed as a sum of few terms.

Some subsets of a grid of points have also remarkable interpolation formulae that resemble univariate ones [4, 8, 10]. In paragraphs 231 and 232 of Chapter 19 of [8], extensions of the Newton formulae to certain subgrids are examined. The lower sets of nodes are subsets of a grid that can be indexed by a lower set of indices. In these sets, the grid structure can be used to identify subgrids where a tensor product construction can be applied on certain subsets in order to reduce the problem to univariate interpolation subproblems. In fact, the Newton formula on the points of a complete grid can be extended to lower sets. Bivariate interpolation problems with nodes concentrated in layers around the boundary arise in the finite element method. A suitable indexing shows that the nodes are indeed in many cases lower sets. Boolean sums of univariate interpolation operators can be used to express the interpolant on lower sets [4]. A recent paper [5] provides some formulae for reducing the interpolation problem on lower sets to interpolation problems on subgrids which can be related with boolean sums. Some practical ideas for computational implementation using representations in terms of Chebyshev polynomials in 2 and 3 variables can be found in [7]. Since the fundamental polynomials on the subgrids can be expressed as a product of linear factors, it seems reasonable that the

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Lagrange fundamental polynomials on lower sets can be expressed as a sum of few terms. In this paper we show that a low number of terms is needed to express the Lagrange fundamental polynomials on lower sets. The number of terms can be related with the number of maximal indices in the lower set.

In Section 2, lower sets are defined and the Newton formula for these sets is recalled. In Section 3, we show that the projections of lower sets and their coordinate level sets can be regarded as lower sets in smaller dimensions. This fact allows us to express the fundamental Lagrange polynomials of a d-variate problem in terms of fundamental Lagrange polynomials in $d-1$ variables. An example is provided to illustrate the recursion. In Section 4, a bivariate Lagrange formula for lower sets is presented and applications to the problem of computing Lebesgue constants are included.

2 Interpolation formulae for lower sets.

Let $\mathbb{N}_0^d$ be the set of all nonnegative multiindices in d variables. Given $\alpha,\beta\in\mathbb{N}_0^d$, we write $\alpha\le\beta$ to indicate that $\alpha_l\le\beta_l$, for all $l=1,\dots,d$. The least multiindex $(0,0,\dots,0)$ will be denoted simply by 0. For any $\alpha\in\mathbb{N}_0^d$, let us denote by

$$B_\alpha:=\{\beta\in\mathbb{N}_0^d\mid\beta\le\alpha\}.$$

We observe that this set is the cartesian product of univariate sets, $B_\alpha=\prod_{l=1}^d B_{\alpha_l}$, where $B_i:=\{0,1,\dots,i\}$, $i\in\mathbb{N}_0$.

We can pose a Lagrange interpolation problem on a grid of points

$$X_\alpha=\prod_{l=1}^d X_{\alpha_l,l}\subset\mathbb{R}^d,\quad X_{\alpha_l,l}:=\{x_{0,l},x_{1,l},\dots,x_{\alpha_l,l}\}.$$

Each element of the grid X_α can be indexed by an element of the set B_α in the following way

$$x_\beta:=(x_{\beta_1,1},\dots,x_{\beta_d,d}),\quad\beta\in B_\alpha.$$

Let us remark that we only assume that the coordinates $x_{0,l},x_{1,l},\dots,x_{\alpha_l,l}$ are distinct but not necessarily ordered. Thus nodes with consecutive indices may not be contiguous in a geometric sense in the grid X_α. Let us denote by $x=(x_1,\dots,x_d)$ the vector of variables and $x^\beta=x_1^{\beta_1}\cdots x_d^{\beta_d}$ the d-variate monomial whose exponents are the components of the multiindex β. Let us define the space of polynomials

$$P_\alpha:=\langle x^\beta:\beta\le\alpha\rangle=P_{\alpha_1}\otimes\cdots\otimes P_{\alpha_l}.$$

It is well-known that the Lagrange interpolation problem on P_α for the set of nodes X_α has a unique solution and the interpolant p of a function f can be expressed by means of a Lagrange formula

$$p(x)=\sum_{\beta\in B_\alpha}f(x_\beta)l_{x_\beta}(x;X_\alpha).$$

The algebraic structure of the tensor product space P_α can be used to show that the Lagrange fundamental polynomials are products of fundamental polynomials in each variable

$$l_{x_\beta}(x_1,\dots,x_d;X_\alpha)=\prod_{l=1}^d l_{x_{\beta_l,l}}(x_l;X_{\alpha_l,l}),\quad\beta\in B_\alpha.$$

If $\alpha_l>0$, we have

$$l_{x_{\beta_l,l}}(x_l;X_{\alpha_l,l})=\prod_{j\ne\beta_l,0\le j\le\alpha_l}\frac{x_l-x_{j,l}}{x_{\beta_l,l}-x_{j,l}},\quad l=1,\dots,d,$$

and if $\alpha_l=0$, $l_{x_{0,l}}(x_l,X_{0,l})=1$. From now on, we use the convention that the product $\prod_{j\ne\beta_l,0\le j\le\alpha_l}(x_l-x_{j,l})/(x_{\beta_l,l}-x_{j,l})$ equals 1 when $\alpha_l=0$.

We can also write a Newton formula for the interpolant. For this purpose we use the well-known tensor product divided differences [6, 9] (see also [5])

$$[X_{\alpha:\beta}]f=[x_{\alpha_1,1},\dots,x_{\beta_1,1};\dots;x_{\alpha_d,d},\dots,x_{\beta_d,d}]f$$

obtained by successive application of the univariate divided differences. We introduce the tensor product Newton basis of the grid X_α

$$\omega_\beta(x_1,\dots,x_d):=\prod_{l=1}^d\omega_{\beta_l,l}(x_l),\quad\beta\in B_\alpha,$$

where

$$\omega_{\beta_l,l}(x_l):=\prod_{0\le j<\beta_l}(x_l-x_{j,l}),\quad l=1,\dots,d.$$

If $\beta_l=0$, the product extended over an empty set of indices means $\omega_{0,l}(x_l):=1$. From the definition, it follows that $\omega_\beta\in P_\beta\subseteq P_\alpha$. Then Newton's formula for the interpolant can be expressed in the following way

$$p(x)=\sum_{\beta\in B_\alpha}[X_{0:\beta}]f\ \omega_\beta(x).$$

The Lebesgue function

$$\lambda(x;X_\alpha) := \sum_{\beta\in B_\alpha} |l_{x_\beta}(x;X_\alpha)|,$$

is a measure of the stability of the interpolating polynomial at x. Its maximum value on a given domain D is called the Lebesgue constant $\Lambda(X_\alpha)$. In the case of grids a natural domain for the polynomial interpolant is $D=[X_\alpha]$, the convex hull of the elements of the grid. From the fact that the Lagrange fundamental polynomials are products of fundamental polynomials in each variable, it follows that

$$\lambda(x_1,\dots,x_d;X_\alpha) = \prod_{l=1}^d \lambda(x_l;X_{\alpha_l,l}),$$

and

$$\Lambda(X_\alpha) = \prod_{l=1}^d \Lambda(X_{\alpha_l,l}),$$

provided that the Lebesgue constants are computed on the convex hull of the corresponding nodes.

We want to extend the usual tensor product interpolation formulae to sets of nodes indexed by subsets of B_α. Associated with any subset of multiindices L, there exists a space of multivariate polynomials

$$P_L := \langle x^\beta : \beta \in L\rangle$$

and a corresponding subset of nodes of the grid X_α

$$X_L := \{x_\beta : \beta \in L\}.$$

Definition 2.1. A set $L \subseteq \mathbb{N}_0^d$ is called a lower set if it contains all multiindices lower than or equal to any $\beta\in L$, that is, $B_\beta \subseteq L$ for any $\beta \in L$.

A block of a lower set is any subset B_β, $\beta\in L$. From the definition, it follows that a lower set is a union of blocks. A *maximal* element of a lower set L is any $\alpha\in L$ such that if $\beta\in L$ satisfies $\alpha\le\beta$, then $\beta=\alpha$. If L is a finite lower set, each block is contained in a maximal block, associated to a maximal multiindex in L. Thus any finite lower set is the union of the blocks B_α, where α is a maximal element of L. The lower set L corresponds to a complete grid, $L=B_\alpha$, if and only if there exists only one maximal element α in L. If L is a lower set of indices, then the set $X_L := \{x_\beta : \beta\in L\}$ is called a lower set of nodes.

In Theorem 2.1 of [5], it was shown that the Lagrange interpolation problem on a lower set of nodes X_L has a unique solution in P_L . If L is a finite lower set, then $\omega_\beta \in P_L$ for any $\beta\in L$. Thus the Newton formula can be extended to lower sets of nodes. We restate this result below.

Theorem 2.1. *Let L be a finite lower set and X_L be a corresponding lower set of nodes. Let f be a function defined on X_L. There exists a unique polinomial $p\in P_L$ such that $p(x_\beta)=f(x_\beta)$ for all $\beta\in L$, given by*

$$p(x) = \sum_{\beta\in L} [X_{0:\beta}]f\ \omega_\beta(x).$$

Our purpose is to provide a formula for the fundamental polynomials (also called the Lagrange polynomials) $l_{x_\beta}(x;X_L)$ of the Lagrange interpolation problem on X_L in P_L associated with the node x_β, uniquely defined by the property

$$l_{x_\beta}(x_\alpha;X_L) = \delta_{\alpha,\beta}, \quad \forall \alpha,\beta\in L.$$

Here $\delta_{\alpha,\beta}$ stands for the usual Kronecker symbol, whose value is 0 if $\alpha\ne\beta$ and 1 if $\alpha=\beta$. This formula should contain as few terms as possible in order to provide a simple Lagrange formula for the interpolant

$$p(x) = \sum_{\beta\in L} f(x_\beta) l_{x_\beta}(x;X_L)$$

and to derive properties of the Lebesgue function

$$\lambda(x;X_L) = \sum_{\beta\in L} |l_{x_\beta}(x;X_L)|.$$

3 A recurrence relation for fundamental polynomials of lower sets.

In the following lemma, we obtain lower sets from a given one by considering all multiindices greater than or equal to a given one.

Lemma 3.1. *Let $L\subseteq \mathbb{N}_0^d$ be a lower set. For any $\beta\in L$, let $L_\beta := \{\alpha\in L \mid \alpha\ge\beta\}$. Then $L_\beta - \beta := \{\alpha-\beta \mid \alpha\in L_\beta\}$ is a lower set.*

Proof. Let $\gamma\in L_\beta-\beta$ and $\eta\in{\mathbb{N}_0}^d$ with $\eta\le\gamma$. Then $\beta+\gamma\in L_\beta\subseteq L$. Since L is a lower set and $\beta+\eta\le\beta+\gamma$, we have that $\beta+\eta\in L$. Furthermore $\beta+\eta\ge\beta$, which implies that $\beta+\eta\in L_\beta$ or equivalently, $\eta\in L_\beta-\beta$. □

Let L be a finite lower set. By Theorem 2.1, the Lagrange interpolation problem on the subset of X_L

$$X_{L_\beta} := \{x_\alpha : \alpha \in L_\beta\}$$

in the subspace $P_{L_\beta-\beta}$ of P_L has a unique solution because it can be associated with the lower set of indices $L_\beta - \beta$. Let $H_{k,l}$ denote the hyperplane with equation $x_l = x_{k,l}$. Then the set X_{L_β} is the subset of X_L obtained by removing the nodes in the hyperplanes $H_{k,l}$, $0 \le k < \beta_l$, $l = 1, \dots, d$,

$$X_{L_\beta} = X_L \setminus \bigcup_{l=1}^{d} \bigcup_{0\le k<\beta_l} H_{k,l}.$$

This observation allows us to relate the interpolation problem in X_L with the interpolation problem in X_{L_β} and describe the fundamental polynomial associated to the node x_β in the set X_L in terms of the fundamental polynomials associated to x_β in the sets X_{L_β} and X_β, respectively.

Proposition 3.2. *Let L be a lower set, then $l_{x_\beta}(x; X_L) = l_{x_\beta}(x; X_{L_\beta}) l_{x_\beta}(x; X_\beta)$.*

Proof. Let $p(x) := l_{x_\beta}(x; X_{L_\beta}) l_{x_\beta}(x; X_\beta)$. Let us observe that p is the product of a polynomial in $P_{L_\beta-\beta}$ and a polynomial in P_β. By the additive property of the partial degrees, we deduce that $p \in P_{L_\beta} \subseteq P_L$. Let us show that $p(x_\alpha) = \delta_{\alpha,\beta}$ for any $\alpha \in L$. If $\alpha = \beta$, then

$$p(x_\beta) = l_{x_\beta}(x_\beta; X_{L_\beta}) l_{x_\beta}(x_\beta; X_\beta) = 1.$$

If $\alpha \in L_\beta$ and $\alpha \neq \beta$, then $0 \neq \alpha - \beta \in L_\beta - \beta$ and $l_{x_\beta}(x_\alpha; X_{L_\beta}) = 0$. Thus we have

$$p(x_\alpha) = l_{x_\beta}(x_\alpha; X_{L_\beta}) l_{x_\beta}(x_\alpha; X_\beta) = 0.$$

Finally, if $\alpha \in L \setminus L_\beta$, then there exists $l \in \{1, \dots, d\}$ such that $\alpha_l < \beta_l$. Thus $l_{x_\beta}(x_\alpha; X_\beta) = 0$, which implies that $p(x_\alpha) = 0$. Therefore p is the fundamental polynomial associated with x_β in X_L. □

Recall that the Lagrange fundamental polynomials $l_{x_\beta}(x; X_\beta)$ on the grid X_β can be expressed as a product of univariate Lagrange polynomials and thus they are explicitly given by

$$l_{x_\beta}(x; X_\beta) = \prod_{l=1}^{d} \prod_{0\le k<\beta_l} \frac{x_l - x_{k,l}}{x_{\beta_l,l} - x_{k,l}}.$$

So, the problem of finding fundamental polynomials of a lower set for any index can be reduced to the problem of finding the fundamental polynomial in any lower set associated with the node corresponding to the lowest index 0.

Now we want to relate interpolation problems on lower sets with interpolation problems on lower sets in less dimensions. Let us introduce the projection mapping

$$\pi_s : (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d \mapsto (\alpha_1, \dots, \alpha_s) \in \mathbb{N}_0^s$$

and the complementary projection

$$\pi'_{d-s} : (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d \mapsto (\alpha_{s+1}, \dots, \alpha_d) \in \mathbb{N}_0^{d-s}.$$

Observe that both projections are nondecreasing mappings, that is, if $\beta \le \alpha$ in L, then $\pi_s(\beta) \le \pi_s(\alpha)$ and $\pi'_{d-s}(\beta) \le \pi'_{d-s}(\alpha)$. If L is a set of indices, and $\eta \in \pi_s(L)$, then π'_{d-s} is a bijective mapping between each level set $\pi_s^{-1}(\eta) \cap L$ and the set of indices

$$L'(\eta) := \pi'_{d-s}(\pi_s^{-1}(\eta) \cap L) = \{(\alpha_{s+1}, \dots, \alpha_d) \in \mathbb{N}_0^{d-s} \mid (\eta_1, \cdots, \eta_s, \alpha_{s+1}, \dots, \alpha_d) \in L\}.$$

Let us show some relevant properties of the projection of lower sets.

Lemma 3.3. *Let L be a lower set.*

(a) The set of indices $\pi_s(L) \subseteq \mathbb{N}_0^s$ is a lower set.

(b) For any $\eta \in \pi_s(L)$, the set $L'(\eta)$ is a lower set.

(c) If $\gamma \le \eta$, then $L'(\eta) \subseteq L'(\gamma)$.

Proof. (a) If $\eta \le \pi_s(\alpha)$, then we have that $(\eta_1, \dots, \eta_s, \alpha_{s+1}, \dots, \alpha_d) \le \alpha$ and since L is a lower set with $\alpha \in L$, we have $(\eta_1, \dots, \eta_s, \alpha_{s+1}, \dots, \alpha_d) \in L$. Therefore $\eta \in \pi_s(L)$ and $\pi_s(L)$ is a lower set.

(b) Let $\alpha \in L'(\eta)$ and assume that $\beta \le \alpha$. Then we have that $(\eta, \alpha) \in L$ and $(\eta, \beta) \le (\eta, \alpha)$. Therefore $(\eta, \beta) \in L$, that is, $\beta \in L'(\eta)$.

(c) Assume that $\gamma \le \eta$ and let $\alpha \in L'(\eta)$. Then $(\alpha, \eta) \in L$. Since L is a lower set, $(\alpha, \gamma) \in L$ and then $\alpha \in L'(\gamma)$. □

We can associate with $\pi_s(L)$ the set of nodes

$$X_{\pi_s(L)} := \{(x_{\eta_1,1}, \dots, x_{\eta_s,s}) \mid \eta \in \pi_s(L)\} \subset \mathbb{R}^s.$$

By Theorem 2.1, the interpolation problem on the set $X_{\pi_s(L)}$ in $P_{\pi_s(L)}$ has a unique solution. Analogously, we associate with each level set $\pi_s^{-1}(\eta) \cap L$, $\eta \in \pi_s(L)$, the set of nodes

$$X_{L'(\eta)} := \{(x_{\alpha_{s+1},s+1}, \dots, x_{\alpha_d,d}) \mid (\alpha_{s+1}, \dots, \alpha_d) \in L'(\eta)\} \subset \mathbb{R}^{n-s}$$

and obtain a unisolvent problem in $P_{L'(\eta)}$.

Let us describe the fundamental polynomials $l_{x_\beta}(x;X_L)$ of the interpolation problem in d variables in terms of the fundamental polynomials associated with the interpolation problems in 1 and $d-1$ variables. According Proposition 3.2, we can remove nodes in hyperplanes corresponding to sufficient low indices and assume without loss of generality that $\beta=0$. We start by projecting the first component to obtain an interpolation problem on the subset $\pi_1(L)$ of the real line. The fundamental polynomial of $x_{0,1}\in X_{\pi_1(L)}$ is of the form

$$l_{x_{0,1}}(x_1;X_{\pi_1(L)})=\prod_{k=1}^{n}\frac{x_1-x_{k,1}}{x_{0,1}-x_{k,1}},$$

where n is the maximal element in $\pi_1(L)=\{0,1,\dots,n\}\subset\mathbb{N}_0$. We shall also use Lagrange polynomials associated with projections of maximal blocks, which are polynomials of the form

$$l_{x_{0,1}}(x_1;X_{i,1})=\prod_{k=1}^{i}\frac{x_1-x_{k,1}}{x_{0,1}-x_{k,1}},$$

where $X_{i,1}:=\{x_{0,1},\dots,x_{i,1}\}$. For each $i\in\pi_1(L)$ we shall also consider the sets $L'(i)\subset\mathbb{N}_0^{d-1}$ and the Lagrange polynomials associated with the corresponding $(d-1)$-dimensional sets of nodes $X_{L'(i)}$. In the next result we express $l_{x_0}(x;X_L)$ in terms of fundamental polynomials in 1 and $d-1$ variables.

Theorem 3.4. *Let L be a finite lower set, $V\in L$ the set of maximal elements of L and $\pi_1(V):=\{\alpha_1\mid\alpha\in V\}$ the set of first indices of the elements in V. Let $\#\pi_1(V)=m+1$ and $i_0<\cdots<i_m$ be the ordered sequence of indices in $\pi_1(V)$. For any j in $\{0,\dots,m\}$, let $X_{L'(i_j)}$ is the set of nodes in $\mathbb{R}^{d-1}$ associated with the set of multiindices*

$$L'(i_j)=\{(\alpha_2,\dots,\alpha_d)\mid(i_j,\alpha_2,\dots,\alpha_d)\in L\}.$$

Then

$$\begin{aligned}l_{x_0}(x_1,\dots,x_d;X_L)&=\sum_{j=0}^{m-1}l_{x_{0,1}}(x_1;X_{i_j,1})\big(l_{(x_{0,2},\dots,x_{0,d})}(x_2,\dots,x_d;X_{L'(i_j)})-l_{(x_{0,2},\dots,x_{0,d})}(x_2,\dots,x_d;X_{L'(i_{j+1})})\big)\\&+\ l_{x_{0,1}}(x_1;X_{i_m,1})l_{(x_{0,2},\dots,x_{0,d})}(x_2,\dots,x_d;X_{L'(i_m)}).\end{aligned}$$

Proof. For the sake of brevity we denote by $x'=(x_2,\dots,x_d)$, so that $x=(x_1,x')$. For any $\alpha\in L$, we use the notation $\alpha'=(\alpha_2,\dots,\alpha_d)$, $x'_\alpha:=(x_{\alpha_2,2},\dots,x_{\alpha_d,d})$, so that $\alpha=(\alpha_1,\alpha')$ and $x_\alpha=(x_{\alpha_1,1},x'_\alpha)$.

Since the interpolation problem has unique solution in X_L, it is sufficient to show that the polynomial

$$p(x):=\sum_{j=0}^{m-1}l_{x_{0,1}}(x_1;X_{i_j,1})(l_{x'_0}(x';X_{L'(i_j)})-l_{x'_0}(x';X_{L'(i_{j+1})}))+l_{x_{0,1}}(x_1;X_{i_m,1})l_{x'_0}(x';X_{L'(i_m)})$$

belongs to P_L and satisfies $p(x_\alpha)=\delta_{\alpha,0}$ for all $\alpha\in L$. The last term is $l_{x_{0,1}}(x_1;X_{i_m,1})l_{x'_0}(x';X_{L'(i_m)})\in P_{i_m}\otimes P_{L'(i_m)}$. So, it can be expressed as a sum of monomial terms of degree (i,β') with $i\le i_m$ and $\beta'\in L'(i_m)$. From the definition of $L'(i_m)$, it follows that $(i_m,\beta')\in L$ and since L is a lower set $(i,\beta')\in L$. So we have seen that

$$l_{x_{0,1}}(x_1;X_{i_m,1})l_{x'_0}(x';X_{L'(i_m)})\in P_L.$$

The other terms are differences of two polynomials. The first one, $l_{x_{0,1}}(x_1;X_{i_j,1})l_{x'_0}(x';X_{L'(i_j)})$, is a linear combination of monomials of degree (i,β'), with $i<i_j$ and $\beta'\in L'(i_j)$. Since L is a lower set and $(i,\beta')\le(i_j,\beta')\in L$, we have that $(i,\beta')\in L$. The second one, $l_{x_{0,1}}(x_1;X_{i_j,1})l_{x'_0}(x';X_{L'(i_{j+1})})$, is a linear combination of monomials of degree (i,β'), with $i<i_j$ and $\beta'\in L'(i_{j+1})$. By Lemma 3.3 (c), $L'(i_{j+1})\subseteq L'(i_j)$ and then $(i,\beta')\in L$. Therefore

$$l_{x_{0,1}}(x_1;X_{i_j,1})(l_{x'_0}(x';X_{L'(i_j)})-l_{x'_0}(x';X_{L'(i_{j+1})}))\in P_L.$$

So we have shown that $p\in P_L$.

If $\alpha_1=0$, we have that $l_{x_{\alpha_1,1}}(x_{0,1};X_{i_j,1})=l_{x_{0,1}}(x_{0,1};X_{i_j,1})=1$ for all $j\in\{0,\dots,m\}$ and then

$$p(x_\alpha)=\sum_{j=1}^{m-1}(l_{x'_0}(x'_\alpha;X_{L'(i_j)})-l_{x'_0}(x'_\alpha;X_{L'(i_{j+1})}))+l_{x'_0}(x'_\alpha;X_{L'(i_m)})=l_{x'_0}(x'_\alpha;X_{L'(i_1)}).$$

Since $l_{x'_0}(x';X_{L'(i_1)})$ is the Lagrange polynomial associated with the node x'_0 in the set $X_{L'(i_1)}$, we have

$$l_{x'_0}(x'_\alpha;X_{L'(i_1)})=\delta_{\alpha,0}.$$

Now assume that $\alpha_1\ne0$ and let

$$S_\alpha:=\{j\in\{0,\dots,m\}\mid\text{there exists }\beta=(i_j,\beta')\in V\text{ with }\alpha\le\beta\}.$$

Since there exists $\beta \in V$ such that $\alpha \le \beta$, the set S_α is nonempty and $s := \min S_\alpha$ is a well-defined number. From the definition it follows that $\alpha_1 \le i_s$. So $l_j(x_{\alpha_1,1}) = 0$ for all $j \ge s$. Then we have that

$$p(x_\alpha) = \sum_{j=0}^{s-1} l_{x_{0,1}}(x_{\alpha_1,1}; X_{i_j,1})(l_{x'_0}(x'_\alpha; X_{L'(i_j)}) - l_{x'_0}(x'_\alpha; X_{L'(i_{j+1})})).$$

From the definition of s, there exists $\beta' \in L'(i_s)$ such that $\alpha' \le \beta'$. By Lemma 3.3 (b) and Lemma 3.3 (c), $L'(i_s)$ is a lower set contained in the lower set $L'(i_j)$ for all $j \le s$. We deduce that $\alpha' \in L'(i_j)$ for all $j \le s$. Then we have that $l_{x'_0}(x'_\alpha; X_{L'(i_j)}) = \delta_{\alpha',0}$ for all $j \le s$ and

$$l_{x'_0}(x'_\alpha; X_{L'(i_j)}) - l_{x'_0}(x'_\alpha; X_{L'(i_{j+1})}) = \delta_{\alpha',0} - \delta_{\alpha',0} = 0, \quad 0 \le j \le s-1.$$

Therefore we have $p(x_\alpha) = 0$. □

Example 3.1. Let us illustrate Theorem 3.4 with an example in $\mathbb{R}^3$. Let us consider the set of indices $L = L_0 \cup L_1$ with

$$L_0 = \{(0,0,0),(0,0,1),(0,0,2),(0,1,0),(0,1,1),(0,1,2),(0,2,0),(0,2,1)\}$$

and

$$L_1 = \{(1,0,0),(1,0,1),(1,1,0),(1,1,1),(1,2,0)\}.$$

We have the set of maximal elements

$$V = \{(0,1,2),(0,2,1),(1,1,1),(1,2,0)\}$$

and the set of ordered first indices of the elements of V, $\pi_1(V) = \{0,1\}$. For the sake of simplicity, let us take the interpolation nodes $X_L = L$ depicted in Figure 3.1.

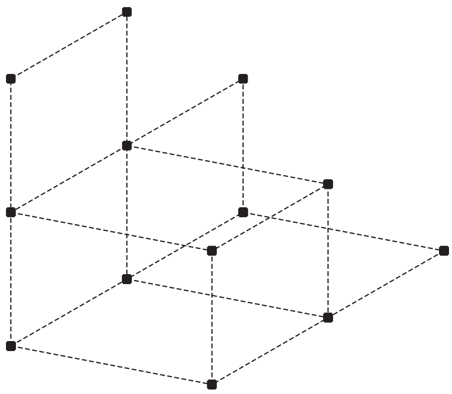

Figure 3.1. A three-dimensional lower set

In this case $X_{L'(0)}$ and $X_{L'(1)}$ are the sets of nodes in $\mathbb{R}^2$ associated respectively with the sets of multiindices

$$L'(0) = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1)\}$$

and

$$L'(1) = \{(0,0),(0,1),(1,0),(1,1),(2,0)\}.$$

So, we have

$$l_0(x_1;0) = 1, \quad l_0(x_1;0,1) = 1 - x_1$$

and

$$l_{(0,0,0)}(x_1,x_2,x_3;X_L) = l_{(0,0)}(x_2,x_3;X_{L'(0)}) - l_{(0,0)}(x_2,x_3;X_{L'(1)}) + (1-x_1) l_{(0,0)}(x_2,x_3;X_{L'(1)}).$$

We can use again Theorem 3.4 to express the bivariate Lagrange functions in terms of univariate Lagrange functions (see the formulae derived in Example 4.2 at the end of Section 4)

$$l_{(0,0)}(x_2,x_3;X_{L'(0)}) = (1-x_2)(1-x_3)\Big(\frac{2-x_2}{2} + \frac{2-x_3}{2} - 1\Big) = \frac{(1-x_2)(1-x_3)(2-x_2-x_3)}{2}$$

and

$$l_{(0,0)}(x_2,x_3;X_{L'(1)}) = (1-x_2)\Big(\frac{2-x_2}{2} + (1-x_3) - 1\Big) = \frac{(1-x_2)(2-x_2-2x_3)}{2}.$$

Then

$$l_{(0,0,0)}(x_1,x_2,x_3;X_L) = l_{(0,0)}(x_2,x_3;X_{L'(0)}) - x_1 l_{(0,0)}(x_2,x_3;X_{L'(1)}) = \frac{(1-x_2)}{2}\left[(1-x_3)(2-x_2-x_3) - x_1(2-x_2-2x_3)\right].$$

4 Fundamental polynomials of bivariate lower sets.

In this section we apply the reduction formula derived in Theorem 3.4 to bivariate lower sets in order to express the bivariate Lagrange polynomials in terms of univariate Lagrange polynomials. We show that the fundamental polynomials are sums involving few terms.

Let n be the number of maximal elements of a bivariate lower set L. Then the set of maximal multiindices can be ordered by increasing order of the first index, that is, $V := \{\alpha^{(j)} \in L \mid j = 0,\dots,m\}$, where $\alpha^{(j)} = (\alpha_1^{(j)},\alpha_2^{(j)})$ and $\alpha_1^{(j)} \le \alpha_1^{(j+1)}$, $j = 0,\dots,m-1$. Observe that if $\alpha_1^{(j)} = \alpha_1^{(j+1)}$ then either $\alpha^{(j)} \le \alpha^{(j+1)}$ or $\alpha^{(j+1)} \le \alpha^{(j)}$. This is a contradiction since both indices are distinct and maximal. Then we have that

$$\alpha_1^{(0)} < \alpha_1^{(1)} < \cdots < \alpha_1^{(m)}.$$

Using Lemma 3.3 (c), we deduce that $\alpha_2^{(j)} \ge \alpha_2^{(j+1)}$. Using the same argument as above, we deduce that

$$\alpha_2^{(0)} > \alpha_2^{(1)} > \cdots > \alpha_2^{(m)}.$$

Now we are ready to apply Theorem 3.4 and derive a formula for the fundamental polynomial corresponding to the index $(0,0)$.

Proposition 4.1. *Let L be a lower set, and $\alpha^{(0)},\dots,\alpha^{(m)}$ be the sequence of maximal multiindices lexicographically ordered. Then we have*

$$l_{x_0}(x_1,x_2;X_L) = \sum_{j=0}^{m-1} l_{x_{0,1}}(x_1;X_{\alpha_1^{(j)},1})\big(l_{x_{0,2}}(x_2;X_{\alpha_2^{(j)},2}) - l_{x_{0,2}}(x_2;X_{\alpha_2^{(j+1)},2})\big) + l_{x_{0,1}}(x_1;X_{\alpha_1^{(m)},1})l_{x_{0,2}}(x_2;X_{\alpha_2^{(m)},2}).$$

In the above formula the polynomials $l_{x_{0,1}}(x_1;X_{\alpha_1^{(j)},1})$ and $l_{x_{0,2}}(x_2;X_{\alpha_2^{(j)},2})$ are given by

$$l_{x_{0,1}}(x_1;X_{\alpha_1^{(j)},1}) = \prod_{k=1}^{\alpha_1^{(j)}} \frac{x_1 - x_{k,1}}{x_{0,1}-x_{k,1}}, \quad l_{x_{0,2}}(x_2;X_{\alpha_2^{(j)},2}) = \prod_{k=1}^{\alpha_2^{(j)}} \frac{x_2 - x_{k,2}}{x_{0,2}-x_{k,2}}, \quad j = 0,\dots,m,$$

and the difference $l_{x_{0,2}}(x_2;X_{\alpha_2^{(j)},2}) - l_{x_{0,2}}(x_2;X_{\alpha_2^{(j+1)},2})$, $j<m$, can be factorized in the following way

$$l_{x_{0,2}}(x_2;X_{\alpha_2^{(j)},2}) - l_{x_{0,2}}(x_2;X_{\alpha_2^{(j+1)},2}) = \prod_{k=1}^{\alpha_2^{(j+1)}} \frac{x_2 - x_{k,2}}{x_{0,2}-x_{k,2}} \Big(\prod_{k=\alpha_2^{(j+1)}+1}^{\alpha_2^{(j)}} \frac{x_2 - x_{k,2}}{x_{0,2}-x_{k,2}} - 1\Big).$$

For any $\beta \le \alpha$, let $B_{\beta:\alpha} := \{\gamma \mid \beta \le \gamma \le \alpha\}$. Then if $L = \bigcup_{j=0}^m B_{\alpha^{(j)}}$ is a lower set and $\beta \in L$, we can write $L_\beta = \bigcup_{j=0}^m B_{\beta:\alpha^{(j)}}$. Let

$$X_{\beta:\alpha} := \{x_\gamma \mid \beta \le \gamma \le \alpha\} = X_{\beta_1:\alpha_1,1} \times X_{\beta_2:\alpha_2,2}$$

then

$$l_{x_\beta}(x_1,x_2;X_{\beta:\alpha}) = l_{x_{\beta_1,1}}(x_1;X_{\beta_1:\alpha_1,1}) l_{x_{\beta_2,2}}(x_2;X_{\beta_2:\alpha_2,2}), \qquad l_{x_{\beta_l,l}}(x_l;X_{\beta_l:\alpha_l,l}) = \prod_{k=\beta_l+1}^{\alpha_l} \frac{x_l - x_{k,l}}{x_{\beta_l,l}-x_{k,l}}, \quad l=1,2. \tag{1}$$

Now we can obtain the following formula for the Lagrange polynomials associated to a given node of a bivariate lower set.

Theorem 4.2. *Let $L \subset \mathbb{N}_0^2$ be a lower set, and $\alpha^{(0)},\dots,\alpha^{(m)}$ be the sequence of maximal multiindices lexicographically ordered. For any $\beta \in L$, the set*

$$J_\beta := \{j \in \{0,\dots,m\} \mid \alpha^{(j)} \ge \beta\}$$

is a set of consecutive indices. Then the Lagrange fundamental polynomials can be expressed by means of the following formula

$$l_{x_\beta}(x_1,x_2;X_L) = l_{x_\beta}(x_1,x_2;X_\beta) \sum_{j\in J_\beta} l_{x_{\beta_1,1}}(x_1;X_{\beta_1:\alpha_1^{(j)},1})\big(l_{x_{\beta_2,2}}(x_2;X_{\beta_2:\alpha_2^{(j)},2}) - l_{x_{\beta_2,2}}(x_2;X_{\beta_2:\alpha_2^{(j+1)},2})\big), \tag{2}$$

where $l_{x_{\beta_l,l}}(x_l;X_{\beta_l:\alpha_l^{(j)},l})$, $l=1,2$, are given by (1) if $j \in J_\beta$ and $l_{x_{\beta_2,2}}(x_l;X_{\beta_2:\alpha^{(j+1)}_2,2})$ denotes the zero polynomial if $j = \max J_\beta$.

Proof. The set of maximal multiindices in L_β is $\alpha^{(j)}$, with $j \in J_\beta$. The indices must be consecutive because $\alpha_1^{(j)}$ is a strictly increasing sequence of integers and $\alpha_2^{(j)}$ is a strictly decreasing sequence of integers. The result follows, combining Proposition 4.1 and Proposition 3.2. □

Let us recall that $l_{x_\beta}(x_1,x_2;X_\beta)$ in (2) can be expressed as a product of linear factors in the following way

$$l_{x_\beta}(x_1,x_2;X_\beta) = \prod_{0\le k<\beta_1} \frac{x_1 - x_{k,1}}{x_{\beta_1,1}-x_{k,1}} \prod_{0\le k<\beta_2} \frac{x_2 - x_{k,2}}{x_{\beta_2,2}-x_{k,2}}.$$

Combining this formula with (1), we can rearrange factors and express formula (2) of Theorem 4.2 in the form

$$l_{x_\beta}(x;X_L) = \sum_{j=\min J_\beta}^{\max J_\beta} l_{x_\beta}(x;X_{\alpha^{(j)}}) - \sum_{j=\min J_\beta}^{\max J_\beta - 1} l_{x_\beta}(x;X_{\alpha^{(j,j+1)}}), \tag{3}$$

where $\alpha^{(j,j+1)} := (\alpha_1^{(j)}, \alpha_2^{(j+1)})$. Note that $X_{\alpha^{(j,j+1)}} = X_{\alpha^{(j)}} \cap X_{\alpha^{(j+1)}}$. This formula can be related with the formulae described in Satz 2.3 of Section 2 of [4] and in Section 5 of [5].

Let us illustrate Theorem 4.2 with a relevant example.

Example 4.1. Assume that the set of indices L is

$$L := \{(\alpha_1, \alpha_2) \mid \alpha_1 + \alpha_2 \le n\}.$$

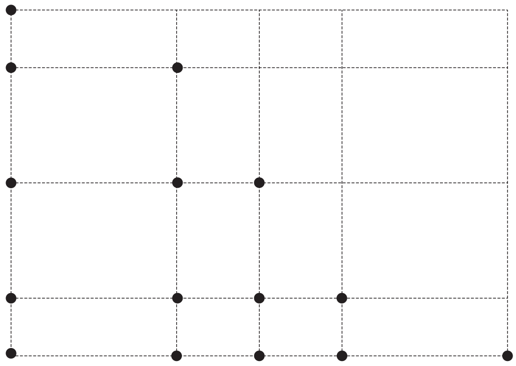

Figure 4.1. A lower set of nodes corresponding to $\alpha_1 + \alpha_2 \le 4$

Then P_L is just P_n^2, the set of bivariate polynomials of total degree less than or equal to n. We observe that the set of maximal indices ordered by the first component is $(j, n-j)$, $j = 0, \dots, n$. Then the set $J_\beta = \{j : \beta_1 \le j \le n - \beta_2\}$ and formula (3) for the fundamental polynomials gives

$$\begin{aligned} l_{x_\beta}(x_1, x_2; X_L) &= \sum_{j=\beta_1}^{n-\beta_2-1} \frac{x_2 - x_{\beta_2,2}}{x_{\beta_2,2} - x_{n-j,2}} \prod_{k\ne\beta_1, 0\le k\le j} \frac{x_1 - x_{k,1}}{x_{\beta_1,1} - x_{k,1}} \prod_{k\ne\beta_2, 0\le k\le n-j-1} \frac{x_2 - x_{k,2}}{x_{\beta_2,2} - x_{k,2}} \\ &+ \prod_{k\ne\beta_1, 0\le k\le n-\beta_2} \frac{x_1 - x_{k,1}}{x_{\beta_1,1} - x_{k,1}} \prod_{k\ne\beta_2, 0\le k\le \beta_2} \frac{x_2 - x_{k,2}}{x_{\beta_2,2} - x_{k,2}}. \end{aligned}$$

The above results allow us to express the fundamental Lagrange polynomials as a sum of few terms and can be applied to the problem of computing Lebesgue constants of bivariate lower sets. Let us show how to bound the Lebesgue function of a bivariate lower set in terms of Lebesgue functions of univariate sets of nodes.

Proposition 4.3. *Let $L \subset \mathbb{N}_0^2$ be a lower set, and $\alpha^{(0)}, \dots, \alpha^{(m)}$ be the sequence of maximal multiindices lexicographically ordered. Then the Lebesgue function of X_L satisfies*

$$\lambda(x_1, x_2; X_L) \le \sum_{j=0}^{m} \lambda(x_1; X_{\alpha_1^{(j)},1}) \lambda(x_2; X_{\alpha_2^{(j)},2}) + \sum_{j=0}^{m-1} \lambda(x_1; X_{\alpha_1^{(j)},1}) \lambda(x_2; X_{\alpha_2^{(j+1)},2}).$$

Proof. From formula (3), we have

$$|l_{x_\beta}(x; X_L)| \le \sum_{j=\min J_\beta}^{\max J_\beta} |l_{x_\beta}(x; X_{\alpha^{(j)}})| + \sum_{j=\min J_\beta}^{\max J_\beta - 1} |l_{x_\beta}(x; X_{\alpha^{(j,j+1)}})|.$$

So we deduce that

$$\begin{aligned} \lambda(x; X_L) &\le \sum_{\beta\in L} \sum_{j=\min J_\beta}^{\max J_\beta} |l_{x_\beta}(x; X_{\alpha^{(j)}})| + \sum_{\beta\in L} \sum_{j=\min J_\beta}^{\max J_\beta - 1} |l_{x_\beta}(x; X_{\alpha^{(j,j+1)}})| \\ &= \sum_{j=0}^{m} \sum_{\beta\le\alpha^{(j)}} |l_{x_\beta}(x; X_{\alpha^{(j)}})| + \sum_{j=0}^{m-1} \sum_{\beta\le\alpha^{(j,j+1)}} |l_{x_\beta}(x; X_{\alpha^{(j,j+1)}})| = \sum_{j=0}^{m} \lambda(x; X_{\alpha^{(j)}}) + \sum_{j=0}^{m-1} \lambda(x; X_{\alpha^{(j,j+1)}}). \end{aligned}$$

The result follows, taking into account that the Lebesgue function of a grid is a product of univariate Lebesgue functions. □

The following example illustrates how to find the fundamental polynomials and bound the Lebesgue function in a simple case

Example 4.2. Let us consider now

$$L := B_{(n,0)} \cup B_{(0,m)} = \{(0,0), (1,0) \dots, (n,0), (0,1), \dots, (0,m)\},$$

then $P_L = \langle 1, x_1, \dots, x_1^n, x_2, \dots, x_2^m \rangle = P_{(n,0)} + P_{(0,m)}$. We observe that the set of maximal indices is $\{(0,m), (n,0)\}$.

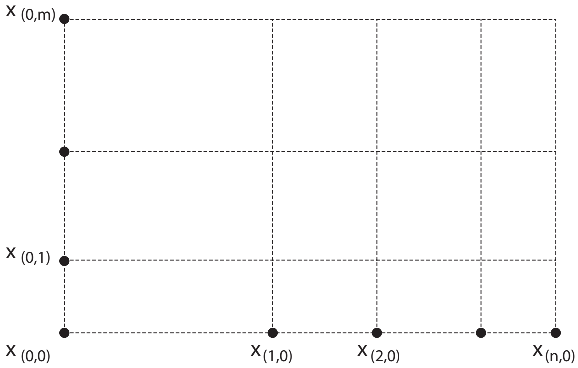

Figure 4.2. A lower set of nodes corresponding to $B_{(n,0)} \cup B_{(0,m)}$

First let us take $\beta = 0$, then we have $J_\beta = \{0, 1\}$. In this simple case, formula (3) gives

$$l_{x_0}(x_1, x_2; X_L) = \prod_{k=1}^{n} \frac{x_1 - x_{k,1}}{x_{0,1} - x_{k,1}} + \prod_{k=1}^{m} \frac{x_2 - x_{k,2}}{x_{0,2} - x_{k,2}} - 1.$$

If $\beta \neq 0$ and $\beta_1 = 0$, only $(0, m) > \beta$ and consequently $J_\beta = \{0\}$. So, the formula for the fundamental polynomials is

$$l_{x_\beta}(x_1, x_2; X_L) = \prod_{k \neq \beta_2; 0 \leq k \leq m} \frac{x_2 - x_{k,2}}{x_{\beta_2,2} - x_{k,2}}.$$

The case $\beta \neq 0$ and $\beta_2 = 0$ is analogous

$$l_{x_\beta}(x_1, x_2; X_L) = \prod_{k \neq \beta_1 0 \leq k \leq n} \frac{x_1 - x_{k,1}}{x_{\beta_1,1} - x_{k,1}}.$$

Finally, as in Proposition 4.3, we can bound the Lebesgue function in terms of the univariate Lebesgue functions in each variable

$$\lambda(x_1, x_2; X_L) \leq \lambda(x_1, X_{n,1}) + \lambda(x_2, X_{m,2}) + 1,$$

and hence the Lebesgue constant on the convex hull of the grid $X_{(n,m)}$ can be bounded by

$$\Lambda(X_L) \leq \Lambda(X_{n,1}) + \Lambda(X_{m,2}) + 1,$$

where $\Lambda(X_{n,1})$ and $\Lambda(X_{m,2})$ are the corresponding univariate Lebesgue constants on the convex hull of the corresponding sets of nodes.

We end showing the growth of the Lebesgue constant for two particular configurations of nodes. We start with the lower set

$$X_L := \{(x_{\alpha_1}, x_{\alpha_2}) | (\alpha_1, \alpha_2) \in L\}, \quad L := \{(\alpha_1, \alpha_2) | \alpha_1 + \alpha_2 \leq n\},$$

subset of $X_n \times X_n$, with $X_n = \{x_0, \ldots, x_n\}$. As mentioned in Example 4.1, the corresponding interpolation space is $P_L = P_n^2$, the set of bivariate polynomials of total degree less than or equal to n. As the set X_n we have taken two different possibilities. The first choice is Chebyshev-Lobatto nodes on $[-1, 1]$

$$x_j = -\cos(j\pi/n), \quad j = 0, \ldots, n.$$

Since the sequence of nodes is increasing, we have

$$x_{\alpha_1} + x_{\alpha_2} \leq x_{n-\alpha_2} + x_{\alpha_2} = \cos(\alpha_2 \pi/n) - \cos(\alpha_2 \pi/n) = 0,$$

and the convex hull of the nodes $[X_L]$ is the triangle with vertices $(-1, -1)$, $(-1, 1)$ and $(1, -1)$. The second choice of X_n are equidistant nodes

$$x_j = -1 + \frac{2j}{n}, \quad j = 0, \ldots, n.$$

Again the convex hull of the nodes is the triangle with vertices $(-1, -1)$, $(-1, 1)$ and $(1, -1)$.

We have computed the Lagrange polynomials on X_L using the formula in Example 4.1 based on Theorem 4.2. The computation of the Lebesgue function is accurate for low degrees. In order to find and approximation of the Lebesgue constant on the triangle we use a sample of points in the convex hull of the form

$$(-1 + 2h_1/N, -1 + 2h_2/N), \quad h_1 + h_2 \leq N.$$

It is hard to find good approximations of the Lebesgue constant because the Lebesgue function is highly oscillating for big values of n. For this reason, we have compared two dense samples with $N = 2048$ and $N = 4000$ and checked that the maximum values agree in the first four digits. In spite of the big values taken for N, we warn about the fact that we cannot ensure that all digits provided in the table are correct, specially for the highest degrees. The following table provides the Lebesgue constants in both cases.

degree	Chebyshev-Lobatto	Equidistant
2	1.6667	1.6667
3	2.9889	2.2698
4	5.7517	3.4748
5	11.490	5.4522
6	25.654	8.7477
7	61.975	14.345
8	158.17	24.008
9	421.27	40.923
10	1152.2	70.892
11	3217.7	124.53

The better behaviour of the lower set with equidistant nodes can be explained by reasons of symmetry and by the fact that the distribution of the subgrid for Chebyshev-Lobatto nodes seems to be not dense enough in the neighbourhood of $(0,0)$. In fact, the maximum value of the Lebesgue function is attained at a point close to the origin (see Figure 4.3 left). In the case of equidistant nodes the maximum value is attained near the vertices (see Figure 4.3 right).

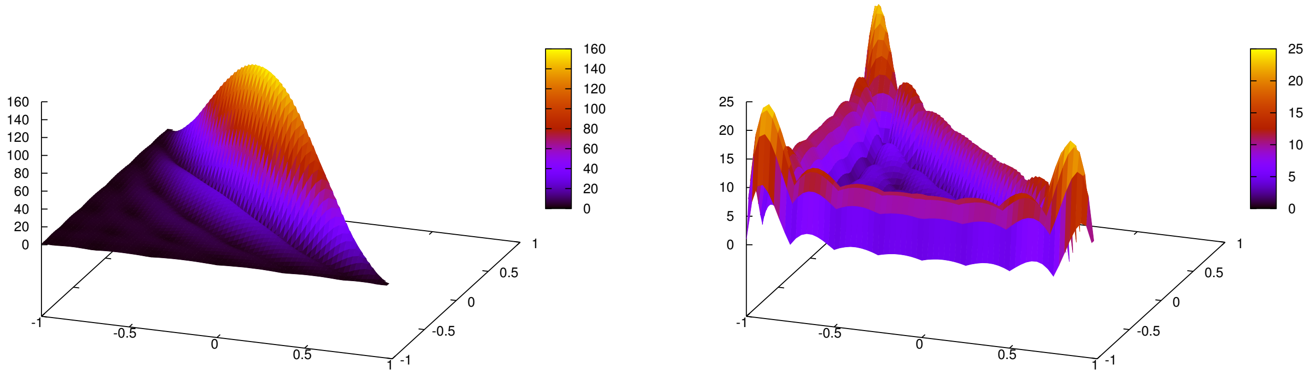

Figure 4.3. Lebesgue functions for degree $n = 8$ on subgrids based on Chebyshev-Lobatto (left) or equidistant nodes (right)

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