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On the Uniqueness of an Orthogonality Property of the Legendre Polynomials

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Abstract

Recently [1] gave a remarkable orthogonality property of the classical Legendre polynomials on the real interval $[-1,1]$: polynomials up to degree n from this family are mutually orthogonal under the arcsine measure weighted by the degree-n normalized Christoffel function. We show that the Legendre polynomials are (essentially) the only orthogonal polynomials with this property.

1 Introduction

Let $\Pi_n(\mathbb{R})$ denote the real univariate polynomials of degree at most n and suppose that μ is a probability measure supported on the interval $[-1,1]$. With the inner-product

$$\langle p,q\rangle := \int_{-1}^1 p(x)q(x)d\mu(x),$$

the Gram-Schmidt process applied to the standard monomial polynomial basis results in a sequence $Q_i(x)$, $i=0,1,2,\cdots$, of *orthonormal* polynomials

$$\langle Q_i,Q_j\rangle = \delta_{ij}.$$

Here, as throughout, we assume that μ is non-degenerate in the sense that if $0\neq p$ is a polynomial, then $\infty > \langle p,p\rangle > 0$.

The reproducing kernel for $\Pi_n(\mathbb{R})$, equipped with this inner-product, is then

$$K_n(x,y) := \sum_{i=0}^n Q_i(x)Q_i(y)$$

and the function

$$\lambda_n(x) := \frac{1}{K_n(x,x)} \tag{1}$$

is known as the associated Christoffel function; it plays an important role in the theory of orthogonal polynomials (see for example the survey article by Nevai [2]).

It is well-known (see e.g. [4]) that

$$\lim_{n\to\infty} \frac{1}{n+1}K_n(x,x)d\mu = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}dx,$$

the latter being the so-called arcsine measure which is also the equilibrium measure of complex potential theory for the interval $[-1,1]$. The convergence is, in general weak$-*$, but in some circumstances even locally uniformly on $(-1,1)$. In other words

$$d\mu = \lim_{n\to\infty} \frac{n+1}{K_n(x,x)}\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}dx$$

or, equivalently,

$$d\mu = \lim_{n\to\infty} (n+1)\lambda_n(x)\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}dx.$$

Hence it would not be totally unexpected that

$$\int_{-1}^1 Q_i(x)Q_j(x)\left[(n+1)\lambda_n(x)\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\right]dx \approx \delta_{ij}, \tag{2}$$

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at least asymptotically.

The result of [1] is that, in the case of $d\mu = (1/2)dx$, so that the orthogonal polynomials $Q_j(x) = P_j^*(x)$, the classical Legendre polynomials suitably orthonormalized, the approximate identity (2) is actually an identity, i.e,

$$\int_{-1}^{1} P_i^*(x)P_j^*(x)\left[(n+1)\lambda_n(x)\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\right]dx = \delta_{ij}, \quad 0 \le i,j \le n. \tag{3}$$

Equivalent identities are

$$\int_{-1}^{1} P_k^*(x)\left[(n+1)\lambda_n(x)\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\right]dx = \delta_{0,k}, \quad 0 \le k \le 2n. \tag{4}$$

and

$$\int_{-1}^{1} p(x)\left[(n+1)\lambda_n(x)\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\right]dx = \int_{-1}^{1} p(x)\frac{1}{2}dx, \quad \deg(p) \le 2n. \tag{5}$$

The purpose of this note is to prove the following uniqueness results:

- Supposing that we have a family of polynomials $\{Q_j\}_{j=0,1,\cdots}$ for which

$$\int_{-1}^{1} Q_k(x)\left[(n+1)\lambda_n(x)\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\right]dx = \delta_{0,k}, \quad 0 \le k \le 2n. \tag{6}$$

  Theorem 2.1 below shows that, already for $n = 1$, the *polynomials* $Q_0(x)$ and $Q_1(x)$ must be the first two (normalized) Legendre polynomials. Further, among all Jacobi *measures* (cf. (7) below) the Legendre case is the only one for which this can be true.

- Theorem 2.4 shows that if (6) holds for $n = 0, 1, \cdots, N$ and the measure $d\mu(x)$ is *symmetric*, then the Q_j, $0 \le j \le N$, must be the (normalized) Legendre polynomials.

- Finally, Theorem 2.5 shows that if we make, instead of symmetry, the assumption that (5) holds up to $k = 2n + 1$ (instead of up just $2n$) then also the Q_j, $0 \le j \le N$, must be the (normalized) Legendre polynomials.

2 Uniqueness Results

The Legendre polynomials are the special case of $\alpha = \beta = 0$ for the family of Jacobi polynomials. It is therefore natural to consider the Jacobi measures

$$d\mu_{\alpha,\beta} = c_{\alpha,\beta}(1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha,\beta > -1, \tag{7}$$

with the constant $c_{\alpha,\beta}$ chosen so that $\mu_{\alpha,\beta}$ is indeed a probability measure.

Theorem 2.1. *Suppose that for some probability measure the associated orthonormal polynomials satisfy*

$$\int_{-1}^{1} Q_k(x)\left[(n+1)\lambda_n(x)\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\right]dx = \delta_{0,k}, \quad 0 \le k \le 2n$$

for $n = 0$ and $n = 1$. Then $Q_0(x) = P_0^*(x) = 1$ and $Q_1(x) = P_1^*(x)$, the normalized Legendre polynomial. In other words, for $n = 1$ the only set of orthogonal polynomials $\{Q_0(x), Q_1(x)\}$ that satisfy the identity is the set of orthonormalized Legendre polynomials $\{P_0^*(x), P_1^*(x)\}$. Further, if the probability measure μ is a Jacobi measure (7), then the only case where $Q_1(x) = P_1^*(x)$ is for $\alpha = \beta = 0$. In other words, already for $n = 1$ the only set of orthogonal Jacobi polynomials that satisfy the identity is in the Legendre case.

Proof of Theorem 2.1. Since we are dealing with a probability measure, $Q_0(x) = 1$. Hence, by assumption we have, for $n = 1$,

$$\frac{2}{\pi}\int_{-1}^{1} \frac{1}{1+Q_1^2(x)}\frac{1}{\sqrt{1-x^2}}dx = 1, \quad (k=0),$$

$$\frac{2}{\pi}\int_{-1}^{1} \frac{Q_1(x)}{1+Q_1^2(x)}\frac{1}{\sqrt{1-x^2}}dx = 0, \quad (k=1).$$

With the substitution $x = \cos(\theta)$ these become

$$\frac{1}{\pi}\int_{0}^{2\pi} \frac{1}{1+Q_1^2(\cos(\theta))}d\theta = 1, \quad (k=0), \tag{8}$$

$$\frac{1}{\pi}\int_{0}^{2\pi} \frac{Q_1(\cos(\theta))}{1+Q_1^2(\cos(\theta))}d\theta = 0, \quad (k=1). \tag{9}$$

Now suppose that $Q_1(x) = ax + b$ for some constants $a \ne 0, b$.

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Lemma 2.2. *Let $\omega := (-b+i)/a$. Then we have*

$$\frac{1}{\pi}\int_0^{2\pi}\frac{1}{1+Q_1^2(\cos(\theta))}d\theta = -\frac{2}{a}\Im\left(\frac{1}{\sqrt{\omega^2-1}}\right) = \frac{2v}{\sqrt{(b^2-a^2-1)^2+4b^2}}$$

where

$$v := \sqrt{\frac{-(b^2-a^2-1)+\sqrt{(b^2-a^2-1)^2+4b^2}}{2}}.$$

Proof. It is easy to verify that the two zeros of $1+(ax+b)^2$ are $x=\omega,\overline{\omega}$. Hence

$$\begin{aligned}\frac{1}{\pi}\int_0^{2\pi}\frac{1}{1+Q_1^2(\cos(\theta))}d\theta &= \frac{1}{\pi}\int_0^{2\pi}\frac{1}{1+(a\cos(\theta)+b)^2}d\theta\\ &= \frac{1}{\pi a^2}\int_0^{2\pi}\frac{1}{(\cos(\theta)-\omega)(\cos(\theta)-\overline{\omega})}d\theta\\ &= \frac{1}{\pi a^2}\frac{1}{\omega-\overline{\omega}}\int_0^{2\pi}\left\{\frac{1}{\cos(\theta)-\omega}-\frac{1}{\cos(\theta)-\overline{\omega}}\right\}d\theta.\end{aligned}$$

But substituting $z=e^{i\theta}$ and converting to a contour integral around the unit circle, one easily sees that

$$\frac{1}{\pi}\int_0^{2\pi}\frac{1}{\cos(\theta)-\omega}d\theta = -\frac{2}{\sqrt{\omega^2-1}}$$

where the branch of the square root is chosen so that $|\omega+\sqrt{\omega^2-1}|>1$. It follows directly then that

$$\frac{1}{\pi}\int_0^{2\pi}\frac{1}{1+Q_1^2(\cos(\theta))}d\theta = -\frac{2}{a}\Im\left(\frac{1}{\sqrt{\omega^2-1}}\right).$$

The rest of the Lemma follows upon confirming that

$$(u+iv)^2 = a^2(\omega^2-1)$$

with v as defined above and $u := -b/v$. □

Lemma 2.3. *With the above notation, we have*

$$\frac{1}{\pi}\int_0^{2\pi}\frac{\cos(\theta)}{1+Q_1^2(\cos(\theta))}d\theta = -2\frac{b}{a}\frac{v^2-1}{v\sqrt{(b^2-a^2-1)^2+4b^2}}.$$

Proof. The proof is elementary, using the same technique as for the previous Lemma. We omit the details. □

From the two Lemmas, the two conditions (8) and (9) may be expressed as:

$$\frac{2v}{\sqrt{D}} = 1, \quad (k=0), \tag{10}$$

$$-2b\frac{v^2-1}{v\sqrt{D}}+b=0, \quad (k=1). \tag{11}$$

where we use the same notation as above for v and have introduced

$$D := (b^2-a^2-1)^2+4b^2.$$

First of all, we claim that $b=0$ for otherwise, if $b\neq 0$, then (11) simplifies to

$$2\frac{v^2-1}{v\sqrt{D}} = 1.$$

Substituting $\sqrt{D}=2v$ (from (10)), then $2(v^2-1)/(2v^2)=1$, but this is clearly not possible. Hence $b=0$, indeed. In this case $D=(a^2+1)^2$, $v=\sqrt{a^2+1}$ and the condition (10) becomes

$$2\frac{\sqrt{a^2+1}}{a^2+1} = 1 \iff a=\pm\sqrt{3},$$

as is easily seen. Since we may assume, with out loss of generality, that $a > 0$, we have $a = \sqrt{3}$ and

$$Q_1(x) = \sqrt{3}x = P_1^*(x).$$

The proof of the Theorem will be completed by verifying that in the Jacobi case $Q_1(x) = P_1^*(x) = \sqrt{3}x$ implies that $\alpha = \beta = 0$. But (see e.g. [3])

$$Q_1(x) = \sqrt{\frac{\alpha+\beta+3}{4(\alpha+1)(\beta+1)}}\{(\alpha+\beta+2)x + (\alpha-\beta)\}.$$

Hence $b = 0$ iff $\alpha = \beta$ in which case

$$Q_1(x) = \sqrt{2\alpha+3}x$$

and $\sqrt{2\alpha+3} = \sqrt{3} \iff \alpha = 0$. □

Theorem 2.4. *Suppose that μ is now a symmetric probability measure (i.e., invariant under $x \to -x$) so that the associated orthonormal polynomials are even or odd according to their degree. Suppose that*

$$\int_{-1}^{1} Q_k(x)\left[(n+1)\lambda_n(x)\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\right]dx = \delta_{0,k}, \quad 0 \le k \le 2n$$

for $n = 0, 1, 2, \cdots, N$. Then

$$Q_j(x) = P_j^*(x), \quad 0 \le j \le N,$$

the orthonormalized Legendre polynomials.

Proof of Theorem 2.4. The case $N = 1$ was done (in more generality) in Theorem 2.1. We proceed by induction. The idea of the proof will be clear already from the the $N = 2$ case. Here $Q_0(x) = P_0^*(x)$ and $Q_1(x) = P_1^*(x) = \sqrt{3}x$. We wish to show that $Q_2(x) = P_2^*(x)$. Now, $K_1(x, x) = 1^2 + (\sqrt{3}x)^2 = 1 + 3x^2$ and so from the $n = 1$ case we must have

$$2\int_{-1}^{1} \frac{Q_2(x)}{1+3x^2}\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}dx = 0.$$

But from the Legendre case we know that

$$2\int_{-1}^{1} \frac{1}{1+3x^2}\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}dx = 1$$

while

$$2 = 2\int_{-1}^{1} \frac{1+3x^2}{1+3x^2}\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}dx = 6\int_{-1}^{1} \frac{x^2}{1+3x^2}\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}dx + 1$$

implies that

$$2\int_{-1}^{1} \frac{x^2}{1+3x^2}\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}dx = \frac{1}{3}.$$

Then writing $Q_2(x) = ax^2 + b$ (it is even by hypothesis) we have

$$0 = 2\int_{-1}^{1} \frac{Q_2(x)}{1+3x^2}\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}dx = a/3 + b$$

so that $b = -a/3$. Consequently

$$Q_2(x) = \frac{a}{3}(3x^2 - 1) = cP_2^*(x)$$

for some constant c, as the Legendre polynomial $P_2(x) = 3x^2 - 1$.

Consequently,

$$K_2(x, x) = 1 + 3x^2 + Q_2^2(x) = 1 + 3x^2 + c^2(P_2^*(x))^2.$$

If now,

$$3\int_{-1}^{1} \frac{1}{K_2(x,x)}\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}dx = 1$$

then by the Legendre case,

$$\begin{aligned} 1 &= 3\int_{-1}^{1} \frac{1}{1+3x^2+(P_2^*(x))^2}\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}dx \\ &= 3\int_{-1}^{1} \frac{1}{1+3x^2+c^2(P_2^*(x))^2}\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}dx. \end{aligned}$$

But if $c^2 > 1$ then $1+3x^2+c^2(P_2^*(x))^2 > 1+3x^2+(P_2^*(x))^2$ (except at a finite set of points) and if $c^2 < 1$ then $1+3x^2+c^2(P_2^*(x))^2 < 1+3x^2+(P_2^*(x))^2$ (except at a finite set of points). Hence we must have $c^2 = 1$ for these two integrals to be equal. It follows that $Q_2(x) = P_2^*(x)$ (the sign is unimportant).

Now for the general case. Suppose then that the Theorem is true for a certain $N \geq 2$. We will show that then it also is true for $N+1$. By the induction hypothesis

$$K_N(x,x) = k_N(x,x) := \sum_{k=0}^{N} (P_k^*(x))^2,$$

the kernel for the Legendre case, and

$$K_{N+1}(x,x) = k_N(x) + Q_{N+1}^2(x).$$

We claim that from our assumptions $Q_{N+1}(x) = cP_{N+1}^*(x)$ for some constant c. To see this just note that by the Gram-Schmidt process

$$Q_{N+1}(x) = C\{x^{N+1} - \sum_{j=0}^{N} \langle x^{N+1}, Q_j(x)\rangle Q_j(x)\}$$

for some normalization constant C. Since x^{N+1} is of opposite parity to $Q_N(x)$,
$\langle x^{N+1}, Q_N(x)\rangle = 0$ and we actually have

$$Q_{N+1}(x) = C\{x^{N+1} - \sum_{j=0}^{N-1} \langle x^{N+1}, Q_j(x)\rangle Q_j(x)\}.$$

But, from the induction hypothesis, $Q_j(x) = P_j^*(x)$, $0 \leq j \leq N$, and so

$$Q_{N+1}(x) = C\{x^{N+1} - \sum_{j=0}^{N-1} \langle x^{N+1}, P_j^*(x)\rangle P_j^*(x)\}.$$

But on the one hand

$$\int_{-1}^{1} Q_k(x)\left[(N+1)\frac{1}{K_N(x.x)}\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\right]dx = \delta_{0,k}, \quad 0 \leq k \leq 2N$$

is equivalent to

$$\int_{-1}^{1} p(x)\left[(N+1)\frac{1}{K_N(x,x)}\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\right]dx = \int_{-1}^{1} p(x)d\mu, \quad \deg(p) \leq 2N$$

while $K_N(x,x) = k_N(x,x)$ informs us that, for $\deg(p) \leq 2N$,

$$\begin{aligned}\int_{-1}^{1} p(x)d\mu &= \int_{-1}^{1} p(x)\left[(N+1)\frac{1}{K_N(x,x)}\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\right]dx \\ &= \int_{-1}^{1} p(x)\left[(N+1)\frac{1}{k_N(x,x)}\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\right]dx \\ &= \int_{-1}^{1} p(x)\frac{1}{2}dx\end{aligned}$$

by the Legendre case. It follows that for $0 \leq j \leq N-1$,

$$\langle x^{N+1}, P_j^*(x)\rangle = \int_{-1}^{1} x^{N+1}P_j^*(x)d\mu = \int_{-1}^{1} x^{N+1}P_j^*(x)\frac{1}{2}dx \tag{12}$$

and hence

$$\begin{aligned}Q_{N+1}(x) &= C\{x^{N+1} - \sum_{j=0}^{N-1}\langle x^{N+1}, P_j^*(x)\rangle P_j^*(x)\} \\ &= C\{x^{N+1} - \sum_{j=0}^{N-1}\langle x^{N+1}, P_j^*(x)\rangle_{\text{legendre}} P_j^*(x)\} \\ &= CP_{N+1}^*(x).\end{aligned}$$

(for a possibility different constant C). The remainder of the argument is exactly as in the $N=1$ case. □

Notice that for a symmetric measure the Christoffel function $\lambda_n(x)$ is an even function. Hence the identity (5) also holds for $p(x) = x^{2n+1}$, both integrals being zero. In particular, for the Legendre case, (5) holds for $\deg(p) \leq 2n+1$. If for a measure μ we assume (5) $\deg(p) \leq 2n+1$, then we also have uniqueness.

Theorem 2.5. *Suppose that μ is a probability measure supported on $[-1,1]$ with the property that*

$$\int_{-1}^{1} Q_k(x)\left[(n+1)\lambda_n(x)\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\right]dx=\delta_{0,k},\quad 0\le k\le 2n+1$$

for $n=0,1,2,\cdots,N$. Then

$$Q_j(x)=P_j^*(x),\quad 0\le j\le N,$$

the orthonormalized Legendre polynomials.

Proof of Theorem 2.5. Just note that, with these assumptions, the inner product formula (12) holds also for $j=N$ and hence we have again $Q_{N+1}(x)=CP_{N+1}^*(x)$. The rest of the argument proceeds as before. □

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