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Global and Local Markov Inequalities in the Complex Plane

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Abstract

We present the current state of the art concerning the global and local Markov inequalities in the complex plane. This paper is based on a talk given during the Workshop on Multivariate Approximation in honor of Prof. Len Bos 60th birthday, and rests on articles [4], [5] and [6].

Our research is inspired by papers by Bos and Milman where global and local Markov inequalities are compared in the real case (see [7], [8]). We are interested in obtaining analogous results in the complex plane in view of further investigation of properties of the Green's function connected with Markov sets.

We first recall two crucial results obtained by Bos and Milman (see [9]).

Theorem A. *Suppose that $E \subset \mathbb{R}^n$ is compact. Then a local Markov inequality with exponent $r \geq 1$ is equivalent to a Geometric inequality with the same exponent $r \geq 1$ and implies a Sobolev inequality (with Whitney norms), also with the same exponent r. A Sobolev inequality in Whitney norm implies a Sobolev inequality in the quotient norm. Conversely, if E admits a Sobolev inequality in the quotient norm with exponent $r \geq 1$, then E admits a local Markov inequality with any exponent $\rho > r$. Moreover, in the regular case, $r = 1$, we may take $\rho = r = 1$.*

Theorem B. *Suppose that $E \subset \mathbb{R}^n$ is compact and $\mathcal{C}^\infty$-determining. Then the following are equivalent:*

1. *E admits a Sobolev type inequality in quotient norm,*
2. *E admits a bounded extension,*
3. *E admits a bounded linear extension,*
4. *E admits a Markov inequality.*

We were intrigued to obtain a corresponding result for sets in the complex plane because of the intricate interconnectedness of multiple distinct global and local properties: Markov inequalities, Kolmogorov (Sobolev) type inequalities, polynomial approximation, extension operators, geometric properties and, ultimately, the behavior of the Green's function, i.e. L-regularity, Hölder continuity and the Łojasiewicz-Siciak inequality. However, a simple adaptation to the complex case of the proof given by Bos and Milman is not possible.

Our goal is to prove or disprove the equivalence of GMI and LMP in the complex plane following the lead of Bos and Milman.

Throughout this paper let E be a compact set in the complex plane.

Definition 1. The set E admits the well known *Global Markov Inequality* GMI(k), where $k \geq 1$, if there exists a constant $M \geq 1$ such that for arbitrary $n \in \mathbb{N}$ and holomorphic polynomial $p \in \mathcal{P}_n$ of degree n we have

$$\|p'\|_E \leq M n^k \|p\|_E$$

where $\|\cdot\|_E$ is the supremum norm on E.

If we take a compact set contained in $\mathbb{R}^n$ and we replace p' by the gradient of p, then we obtain the Markov inequality considered by Bos and Milman.

The next definition was inspired by the local inequality considered in theorems A and B.

Definition 2. A compact set $E \subset \mathbb{C}$ admits the *Local Markov Property* LMP(m), where $m \geq 1$, if there exist constants $c, k \geq 1$ such that

$$\forall n \in \mathbb{N} \quad \forall z_0 \in E \quad \forall 0 < r \leq 1 \quad \forall p \in \mathcal{P}_n \quad \forall j = 1, \ldots, n: \qquad |p^{(j)}(z_0)| \leq \left(\frac{cn^k}{r^m}\right)^j \|p\|_{E \cap B(z_0,r)}.$$

It is evident that the Local Markov Property implies the Global Markov Inequality.

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In [3] we proved

Theorem 3. LMP *implies L-regularity, i.e. the continuity of the Green's function of the unbounded component of the complement of E to the complex plane with logarithmic pole at infinity.*

We will use the following notations. Let

- $E_\delta := \{z \in \mathbb{C} \,:\, \operatorname{dist}(z,E) \le \delta\}$,
- $\mathcal{A}^\infty(E) := \left\{f \in \mathcal{C}^\infty(\mathbb{C}) : \frac{\partial f}{\partial \bar z} \text{ is flat on } E\right\}$ be the family of smooth functions that are $\bar\partial$-flat on the set E,
- $\mathcal{H}^\infty(E) := \left\{f \in \mathcal{C}^\infty(\mathbb{C}) : \frac{\partial f}{\partial \bar z} \equiv 0 \text{ in an open neighborhood of } E\right\}$ be the family of smooth functions that are holomorphic in some open neighborhood of the set E,
- $|f|_{E,\ell} := \sum_{\alpha \in \mathbb{N}_0^2,\ |\alpha|=\ell} \|D^\alpha f\|_E, \qquad \|f\|_{E,\ell} := \|f\|_E + |f|_{E,\ell}$ be regular supremum norms for $f \in \mathcal{C}^\infty(E)$ and $\ell \in \mathbb{N}$,
- $\left[\!\left| f \right|\!\right]_{E,\ell} := \inf\left\{\|\tilde f\|_{\operatorname{conv} E,\ell} : \tilde f \in \mathcal{A}^\infty(E),\ \tilde f_{|E} \equiv f_{|E}\right\}$ be quotient norms for $f \in \mathcal{A}^\infty(E)$ and $\ell \in \mathbb{N}$,
- $\langle\!\langle f \rangle\!\rangle_{E,\ell} := \inf\left\{\|\tilde f\|_{\operatorname{conv} E,\ell} : \tilde f \in \mathcal{H}^\infty(E),\ \tilde f_{|E} \equiv f_{|E}\right\}$ be holomorphic quotient norms for $f \in \mathcal{H}^\infty(E)$ and $\ell \in \mathbb{N}$.

The definition given below has been considered in the real case by Bos and Milman (referred to by them as a Sobolev-Gagliardo-Nirenberg inequality).

Definition 4. A compact set $E \subset \mathbb{C}$ admits the *Kolmogorov Property in Quotient norms* KPQ(m), where $m \ge 1$, if there exist constants $c,k \ge 1$ such that

$$\forall \ell \in \mathbb{N} \quad \forall j \in \mathbb{N} \text{ such that } \ell \ge mj \quad \forall f \in \mathcal{A}^\infty(E) \quad : \qquad |f|_{E,j} \le (c\ell^k)^j \cdot \|f\|_E^{1-\frac{mj}{\ell}} \cdot \left[\!\left| f \right|\!\right]_{E,\ell}^{\frac{mj}{\ell}}.$$

Applying the proof given by Bos and Milman to the complex case we can easily obtain the following

Theorem 5. *For any compact set $E \subset \mathbb{C}$ and $m \ge 1$ we have*

$$\text{LMP}(m) \implies \text{KPQ}(m).$$

In order to obtain a property more convenient for our purpose, we consider two Kolmogorov inequalities for holomorphic functions.

Definition 6. (see [4]) A compact set $E \subset \mathbb{C}$ admits the *Kolmogorov Property in Quotient norms for Holomorphic functions* KPQH(m), where $m \ge 1$, if there exist constants $c,k \ge 1$ such that

$$\forall \ell \in \mathbb{N} \quad \forall j \in \mathbb{N} \text{ such that } \ell \ge mj \quad \forall 0<\delta\le 1 \quad \forall f \in \mathcal{H}^\infty(E_\delta) \ : \qquad |f|_{E,j} \le (c\ell^k)^j \cdot \|f\|_E^{1-\frac{mj}{\ell}} \cdot \langle\!\langle f \rangle\!\rangle_{E_\delta,\ell}^{\frac{mj}{\ell}}.$$

Definition 7. (see [4]) A compact set $E \subset \mathbb{C}$ admits the *Kolmogorov Property for Holomorphic functions* KPH(s), where $s \ge 1$, if there exist constants $c,m,k \ge 1$ such that

$$\forall \ell \in \mathbb{N} \quad \forall j \in \mathbb{N} \text{ such that } \ell \ge mj \quad \forall 0<\delta\le 1 \quad \forall f \in \mathcal{H}^\infty(E_\delta) \ : \qquad |f|_{E,j} \le \left(\frac{c\ell^k}{\delta^s}\right)^{j+c} \cdot \|f\|_E^{1-\frac{mj}{\ell}} \cdot \|f\|_{E_\delta}^{\frac{mj}{\ell}}.$$

We proved in [4] two results analogous to the implication *1.* ⇒ *4.* in theorem B.

Theorem 8. *For any compact set $E \subset \mathbb{C}$ and $m \ge 1$ we have*

$$\text{KPQ}(m) \implies \text{KPQH}(m) \implies \text{KPH}(m).$$

Theorem 9. *For any compact set $E \subset \mathbb{C}$ and $m > s \ge 1$ we have*

$$\text{KPH}(s) \implies \text{LMP}(m).$$

In this fashion we obtained the equivalence of LMP, KPQ, KPQH and KPH, which is the first step of the equivalence of the GMI and LMP in the complex plane (under a necessary assumption):

$$\begin{array}{ccc} & & \text{L-reg.} \\ & & \Uparrow \\ \text{KPQ} & \Longleftarrow & \text{LMP} \\ \Downarrow & & \Uparrow \\ \text{KPQH} & \Longrightarrow & \text{KPH} \end{array}$$

Now we will present the equivalence between the Global Markov Inequality, an Extension Property (EXP) and a Kolmogorov Property in Jackson norms (KPJ). These two notions concern functions of the class s(E), which can be rapidly approximated by holomorphic polynomials.

Let

- $\operatorname{dist}_E(f,\mathcal{P}_n) := \inf\limits_{p\in\mathcal{P}_n} \|f-p\|_E$ denote the error of approximation by holomorphic polynomials,
- $s(E) := \left\{ f \in \mathcal{C}(E) : \ \forall \ell \in \mathbb{N} \ \lim\limits_{n\to\infty} n^\ell \operatorname{dist}_E(f,\mathcal{P}_n) = 0 \right\}$ denote the family of functions on the set E which can be rapidly approximated by holomorphic polynomials,
- $|f|_\ell := \|f\|_E + \sup\limits_{n\in\mathbb{N}} n^\ell \operatorname{dist}_E(f,\mathcal{P}_n)$ denote Jackson norms for $f \in s(E)$ and $\ell \geq 0$.

Definition 10. (see [6]) A compact set $E \subset \mathbb{C}$ admits the *Extension Property* EXP(k), where $k \geq 1$, if it is $\mathcal{A}^\infty$-determining and there exist constants $c, u \geq 1$ such that for all $f \in s(E)$ there exists an extension $\widetilde{f} \in \mathcal{A}^\infty(E)$ with the following properties:

$$\begin{array}{ll} (a) & \widetilde{f}_{|E} \equiv f, \\ (b) & \|\widetilde{f}\|_{\mathbb{C}} \leq c\|f\|_E, \\ (c) & \|\widetilde{f}\|_{\mathbb{C},\ell} \leq (c\ell^u)^{\ell+c}\, |f|_{k\ell+c} \quad \text{for all } \ell \in \mathbb{N}. \end{array}$$

Originally Pleśniak (see [14]), building on earlier joint work with Pawłucki (see [12], [13]), proved that in real space the Global Markov Inequality is equivalent to the existence of a continuous linear operator from the function space $\mathcal{C}^\infty(E)$ with a topology determined by Jackson norms to $\mathcal{C}^\infty(\mathbb{R}^N)$ with the natural topology. Bos and Milman adapted Pleśniak's proof to obtain a bounded extension of $\mathcal{C}^\infty$ functions with homogeneous linear loss of differentiability in the quotient topology, however at the expense of the linearity of the extension operator. We in turn modified their definition of the extension property so that it can be deduced from the Global Markov Inequality for any polynomially convex compact subset of the complex plane. In this definition we replaced the quotient norms with Jackson norms as they work well for functions that are holomorphic in some open neighbourhood of a polynomially convex compact set.

Definition 11. (see [6]) A compact set $E \subset \mathbb{C}$ admits the *Kolmogorov Property in Jackson norms* KPJ(k), where $k \geq 1$, if there exist constants $c, u \geq 1$ such that

$$\forall \ell \in \mathbb{N} \quad \forall j \in \mathbb{N} \text{ such that } \ell \geq j \quad \forall f \in \mathcal{A}^\infty(E) \text{ such that } f_{|E} \in s(E) : \qquad |f|_{E,j} \leq (c\ell^u)^{j+c} \cdot \|f\|_E^{1-\frac{j}{\ell}} \cdot |f_{|E}|_{k\ell+c}^{\frac{j}{\ell}}.$$

Theorem 12. (see [6]) *For any polynomially convex compact set $E \subset \mathbb{C}$ and $k' > k \geq 1$ we have*

$$\mathrm{GMI}(k) \implies \mathrm{EXP}(k+1),$$

$$\mathrm{EXP}(k) \implies \mathrm{KPJ}(k),$$

$$\mathrm{KPJ}(k) \implies \mathrm{GMI}(k').$$

Theorem 12 is the second step of the equivalence of the GMI and LMP in the complex plane:

$$\begin{array}{ccccccc} & & \text{L-reg.} & \Longleftarrow & \text{HCP} & & \\ & & \Uparrow & & \Downarrow & & \\ \text{KPQ} & \Longleftarrow & \text{LMP} & \Longrightarrow & \text{GMI} & \Longrightarrow & \text{EXP} \\ \Downarrow & & \Uparrow & & \Uparrow & \swarrow & \\ \text{KPQH} & \Longrightarrow & \text{KPH} & & \text{KPJ} & & \end{array}$$

Now it remains to investigate the implication KPJ $\Rightarrow$ KPH which would give the equivalence of GMI and LMP in the complex plane. To this end, we introduce the following Jackson property.

Definition 13. (see [5]) A compact set $E \subset \mathbb{C}$ admits the *Jackson Property* JP(s), where $s \geq 1$, if $\mathcal{H}^\infty(E)_{|E} \subset s(E)$ and there exist constants $c, v \geq 1$ such that

$$\forall \ell \in \mathbb{N} \quad \forall 0 < \delta \leq 1 \quad \forall f \in \mathcal{H}^\infty(E_\delta) : \quad |f_{|E}|_\ell \leq \left(\frac{c\ell^v}{\delta^s} \right)^{\ell+c} \cdot \|f\|_{E_\delta}.$$

Theorem 14. (see [6]) *For any $k, s \geq 1$ and any polynomially convex compact set $E \subset \mathbb{C}$ admitting JP(s) we have*

$$\mathrm{KPJ}(k) \implies \mathrm{KPH}(ks).$$

Corollary 15. (see [6]) *For any polynomially convex compact set $E \subset \mathbb{C}$ admitting* JP(s) *we have*

$$\text{GMI} \iff \text{LMP}\,.$$

In order to prove that an additional assumption is necessary in the above equivalence, consider onion sets constructed as follows. For each $j \in \mathbb{N}$ let there be an angle $\varphi_j \in (0, 2\pi)$ and radius $a_j \in (0, 1]$ such that $a_{j+1} < a_j$ and $\lim_{j\to\infty} a_j = 0$. Put $E := \{0\} \cup \bigcup_{j=1}^{\infty} C_j$, where $C_j := \{a_j e^{ti} : \varphi_j \le t \le 2\pi\}$.

Lemma 16. (see [6]) *If the onion set E admits* LMP(m)*, then there exists a constant $c > 0$ such that for all $j \in \mathbb{N}$ we have $a_{j+1} \ge c a_j^m$. If there exists a constant $k > 0$ such that $\varphi_j \le a_{j+1}^{2/k}$ for all $j \in \mathbb{N}$, then the onion set E admits* GMI($k+2$).

Example 17. (see [6]) Fix arbitrarily $a_1 < 1$. If for all $j \in \mathbb{N}$ we let $a_{j+1} := a_j^{j+1}$ then, regardless of the choice of the angles $\{\varphi_j\}_{j\in\mathbb{N}}$, the associated onion set does not admit LMP. However, if we choose sufficiently small angles such that for all $j \in \mathbb{N}$ we have $\varphi_j \le a_{j+1}^{2/k}$, where $k > 0$ is fixed, then the onion set admits GMI($k+2$).

Corollary 15 implies that this example of an onion set does not admit the Jackson Property either. This begs the question which sets do admit this property? We give a partial but highly relevant answer to this question in terms of the behavior of the Green's function.

Definition 18. (see [11]) A compact set $E \subset \mathbb{C}$ admits the *Łojasiewicz-Siciak inequality* ŁS(s), where $s \ge 1$, if there exists a constant $M > 0$ such that

$$\forall z \in E_1 \;:\; g_E(z) \ge M\,\mathrm{dist}(z, E)^s.$$

The Łojasiewicz-Siciak inequality is the opposite of the well known Hölder Continuity Property, which gives an upper bound of the Green's function (see e.g. [10], [1], [15]).

Definition 19. A compact set $E \subset \mathbb{C}$ admits the *Hölder Continuity Property* HCP(k), where $k \ge 1$, if there exists a constant $M \ge 1$ such that

$$\forall z \in E_1 \;:\; g_E(z) \le M\,\mathrm{dist}(z, E)^{1/k}.$$

It is well known that HCP(k) implies GMI(k) for any $k \ge 1$, but the converse implication remains an open problem (compare with [2]).

There exists a strong connection between the Jackson Property and the rate of growth of the Green's function.

Theorem 20. (see [5]) Let $s' > s \ge 1$. Any polynomially convex compact set $E \subset \mathbb{C}$ admitting ŁS(s) and HCP, admits JP(s). Moreover, any compact set $E \subset \mathbb{C}$ admitting JP(s), admits ŁS(s').

The classical Jackson inequality implies:

Theorem 21. *Every compact set $E \subset \mathbb{R} \subset \mathbb{C}$ admits* JP(1).

Taking into account the above theorem, we can see that corollary 15 is a generalization of theorem B by Bos and Milman because if we take a set $E \subset \mathbb{R} \subset \mathbb{C}$ we obtain equivalence without any additional assumption as in theorem B.

We close this paper by offering some questions and problems for further research:

- Does ŁS (plus GMI if necessary) imply JP?
- Does LMP (plus ŁS if necessary) imply HCP?

Totik's Wiener type criterion [10] implies HCP, however the converse implication requires an additional assumption: a (geometric) cone condition or a quantitative (capacity) condition. We conjecture that:

- the cone condition implies ŁS,
- the quantitative condition implies ŁS,
- HCP in conjunction with ŁS implies Totik's criterion.

If so, then lower (HCP) plus upper (ŁS) bounds for capacities imply LMP.

References

[1] V. V. Andrievskii. Constructive function theory on sets of the complex plane through potential theory and geometric function theory. *Surv. Approx. Theory*, 2:1-52, 2006.

[2] M. Baran, L. Bialas-Ciez. Hölder continuity of the Green function and Markov brothers' inequality. *Constr. Approx.*, 40:121-140, 2014.

[3] L. Bialas-Ciez, R. Eggink. L-regularity of Markov sets and of m-perfect sets in the complex plane. *Constr. Approx.*, 27:237-252, 2008.

[4] L. Bialas-Ciez, R. Eggink. Equivalence of the Local Markov Inequality and a Kolmogorov type inequality in the complex plane. *Potential Anal.*, 38:299-317, 2013.

[5] L. Bialas-Ciez, R. Eggink. Jackson's inequality in the complex plane and the Łojasiewicz-Siciak inequality of Green's function. In preparation.

[6] L. Bialas-Ciez, R. Eggink. Equivalence of the Global and Local Markov Inequalities in the Complex Plane. In preparation.

[7] L. Bos, P. Milman. Sobolev-Gagliardo-Nirenberg and Markov type inequalities on subanalytic domains. *Geom. Funct. Anal.*, 5:853-923, 1995.

[8] L. Bos, P. Milman. On Markov and Sobolev type inequalities on sets in $\mathbb{R}^n$. *Topics in Polynomials of One and Several Variables and Their Applications*, World Sci. Publishing, River Edge, NJ, 81-100, 1993.

[9] L. Bos, P. Milman. A Geometric Interpretation and the Equality of Exponents in Markov and Gagliardo-Nirenberg (Sobolev) Type Inequalities for Singular Compact Domains. Preprint.

[10] L. Carleson, V. Totik. Hölder continuity of Green's functions. *Acta Sci. Math., Szeged*, 70:557-608, 2004.

[11] L. Gendre. Inégalités de Markov singulières et approximation des fonctions holomorphes de la classe M, dissertation, Université Toulouse III Paul Sabatier, 2005.

[12] W. Pawłucki, W. Pleśniak. Markov's inequality and $\mathcal{C}^\infty$ functions on sets with polynomial cusps. *Math. Ann.*, 275:467-480, 1986.

[13] W.P awłucki, W. Pleśniak. Extension of $\mathcal{C}^\infty$ functions from sets with polynomial cusps. *Studia Math.*, 88:279-287, 1988.

[14] W. Pleśniak. Markov's inequality and the existence of an extension operator for C^∞ functions. *J. Approx. Theory*, 61:106-117, 1990.

[15] T. Ransford, J. Rostand. Hölder exponents of Green's functions of Cantor sets. *Comput. Methods Funct. Theory*, 8:151-158, 2008.