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A new class of degenerate biparametric Apostol–type polynomials

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Abstract

The main objective of this paper is to introduce and explore two novel classes of degenerate biparametric Apostol-type polynomials, which are based on a definition of degenerate Apostol-type polynomials provided by Subuhi Khan et al. We derive various algebraic and differential properties associated with these polynomials. Additionally, we provide a series of illustrative examples for these newly introduced polynomial families along with their corresponding graphs. The majority of the results are proven utilizing well-established generating functions and identities.

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1 Introduction

Throughout this paper, we use the standard notions: $\mathbb{N} := \{1, 2, \ldots\}$; $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$; $\mathbb{Z}$ denotes the set of integers; $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. Further let $k \in \mathbb{Z}$ and $\lambda, \mu \in \mathbb{C}$.

Recently several authors have been introduced new extensions and modifications of the classical Apostol-Bernoulli, Apostol-Euler, and Apostol-Genocchi polynomials, as well as the family of degenerate polynomials in two variables, see for example [1, 2, 6, 7, 10, 15, 12, 13, 9, 17, 8, 3, 20, 21].

The degenerate cosine-Euler polynomials and degenerate sine-Euler polynomials respectively by (see, [8]):

$$\sum_{n=0}^{\infty} \mathcal{E}^{c}_{n,a}(x,y)\frac{t^n}{n!} = \left(\frac{2}{\lambda(1+at)^{\frac{1}{a}}+1}\right)(1+at)^{\frac{x}{a}}\cos\left(\frac{y}{a}\ln(1+at)\right),$$
$$\sum_{n=0}^{\infty} \mathcal{E}^{s}_{n,a}(x,y)\frac{t^n}{n!} = \left(\frac{2}{\lambda(1+at)^{\frac{1}{a}}+1}\right)(1+at)^{\frac{x}{a}}\sin\left(\frac{y}{a}\ln(1+at)\right).$$

On the other hand,

$$\cos\left(\frac{y}{a}\ln(1+at)\right) = \sum_{m=0}^{\infty}\left(\sum_{k=0}^{[\frac{m}{2}]} a^{m-2k}(-1)^k y^{2k} s(m,2k)\right)\frac{t^m}{m!}, \tag{1}$$

$$\sin\left(\frac{y}{a}\ln(1+at)\right) = \sum_{m=0}^{\infty}\left(\sum_{k=0}^{[\frac{m-1}{2}]} a^{m-2k-1}(-1)^k y^{2k+1} s(m,2k+1)\right)\frac{t^m}{m!}. \tag{2}$$

Since, $(1+at)^{\frac{x}{a}} \to e^{xt}$, $\cos\left(\frac{y}{a}\ln(1+at)\right) \to \cos(yt)$ and $\sin\left(\frac{y}{a}\ln(1+at)\right) \to \sin(yt)$ as $a \to 0$, it is evident that the polynomials obtained were those introduced in the year 2018 (see, [11]).

The first-class Stirling number $s(n,k)$ is given by the following generating function (see, [16]):

$$\frac{1}{k!}[\ln(1+t)]^k = \sum_{n=k}^{\infty} s(n,k)\frac{t^n}{n!}.$$

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The generalized falling factorial $(x|\alpha)_n$ with increment α is defined by (see, [17, Definition 2.3]):

$$(x|\alpha)_n = \prod_{k=0}^{n-1}(x-\alpha k),$$

for positive integer n, with the convention $(x|\alpha)_0 = 1$, it follows that

$$(x|\alpha)_n = \sum_{k=0}^{n} s(n,k)\alpha^{n-k}x^k.$$

From Binomial theorem, we have

$$(1+\alpha t)^{\frac{x}{\alpha}} = \sum_{n=0}^{\infty}(x|\alpha)_n\frac{t^n}{n!}.$$

Next, the well-known families of degenerate Bernoulli polynomials $\mathcal{B}_n(x;a)$, $\mathcal{E}_n(x;a)$ and Genocchi $\mathcal{G}_n(x;a)$ are presented, with parameter $a \in \mathbb{R}$ in the variable x and in a neighborhood centered on the point $t = t_o$, through its generating functions, (see, [4, 5]):

$$\frac{t}{(1+at)^{\frac{1}{a}}-1}(1+at)^{\frac{x}{a}} = \sum_{n=0}^{\infty}\mathcal{B}_n(x;a)\frac{t^n}{n!}. \tag{3}$$

When $x=0$, $\mathcal{B}_n(a) := \mathcal{B}_n(0;a)$ are the corresponding degenerate Bernoulli numbers. It is to be noted from (3) that

$$\lim_{a\to 0}\mathcal{B}_n(x;a) = B_n(x), \quad n \geq 0,$$

where $B_n(x)$ are the $n-$th order Bernoulli polynomials (see, [14]).

$$\frac{2}{(1+at)^{\frac{1}{a}}+1}(1+at)^{\frac{x}{a}} = \sum_{n=0}^{\infty}\mathcal{E}_n(x;a)\frac{t^n}{n!}. \tag{4}$$

For $x=0$, $\mathcal{E}_n(a) := \mathcal{E}_n(0;a)$ are the corresponding degenerate Euler numbers. It follows from (4) that:

$$\lim_{a\to 0}\mathcal{E}_n(x;a)(x;a) = E_n(x), \quad n \geq 0,$$

where $E_n(x)$ are the $n-$th order ordinary Euler polynomials (cf. [14]).

$$\frac{2t}{(1+at)^{\frac{1}{a}}+1}(1+at)^{\frac{x}{a}} = \sum_{n=0}^{\infty}\mathcal{G}_n(x;a)\frac{t^n}{n!}. \tag{5}$$

When $x=0$, $\mathcal{G}_n(a) := \mathcal{G}_n(0;a)$ are the corresponding degenerate Genocchi numbers. Consequently from (5), we have:

$$\lim_{a\to 0}\mathcal{G}_n(x;a) = G_n(x), \quad n \geq 0,$$

where $G_n(x)$ are the $n-$th order ordinary Genocchi polynomials (see, [15]).

Waseem A. Khan introduced the degenerate Hermite–Bernoulli Numbers and polynomials of the second Kind by means of the following generating function (see, [18]):

$$\frac{\log(1+at)^{\frac{1}{a}}}{(1+at)^{\frac{1}{a}}-1}(1+at)^{\frac{x}{a}}(1+at^2)^{\frac{y}{a}} = \sum_{n=0}^{\infty}{}_H\mathcal{B}_n(x;y;a)\frac{t^n}{n!}.$$

For $\lambda, u \in \mathbb{C}$ and $\alpha \in \mathbb{N}$ with $u \neq 1$ the generalized degenerate Apostol–type Frobenius Euler–Hermite polynomials of order α which are given by generating function (see, [19, P. 569]):

$$\left(\frac{1-u}{\lambda(1+at)^{\frac{1}{a}}-u}\right)^{\alpha}(1+at)^{\frac{x}{a}}(1+at^2)^{\frac{y}{a}} = \sum_{n=0}^{\infty}{}_Hh_n(x;y;a;\lambda;u)\frac{t^n}{n!}. \tag{6}$$

Taking $u=-1$ and $\alpha=1$ in (6), we obtain the degenerate Hermite–Euler Polynomials

$$\frac{2}{\lambda(1+at)^{\frac{1}{a}}+1}(1+at)^{\frac{x}{a}}(1+at^2)^{\frac{y}{a}} = \sum_{n=0}^{\infty}{}_H\mathcal{E}_n(x;y;a;\lambda)\frac{t^n}{n!}.$$

On the other hand, Subuhi Khan et. al., introduced and studied the degenerate Apostol-type polynomials order α, denoted by $\mathcal{P}^{(\alpha)}(x;,a;\lambda;\mu;\nu)$ by means of the following generating function (see, [17]):

$$\left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}} = \sum_{n=0}^{\infty}\mathcal{P}_n^{(\alpha)}(x;a;\lambda;\mu;\nu)\frac{t^n}{n!}, \tag{7}$$

where $x \in \mathbb{R}$, $\lambda, \mu, \nu \in \mathbb{C}$; $n \in \mathbb{N}_0$.

For $x = 0$, $\mathcal{P}_n^{(\alpha)}(a;\lambda;\mu;\nu) := \mathcal{P}_n^{(\alpha)}(0;a;\lambda;\mu;\nu)$ denotes the corresponding the degenerate Apostol-type numbers of order α and are defined as:

$$\left(\frac{2^\mu t^\nu}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^\alpha = \sum_{n=0}^{\infty} \mathcal{P}_n^{(\alpha)}(a;\lambda;\mu;\nu)\frac{t^n}{n!}.$$

In view of (7), it follows that

$$\lim_{a\to 0} \mathcal{P}^{(\alpha)}(x;,a;\lambda;\mu;\nu) = \mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu), \quad n \geq 0,$$

where $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)$ are the Apostol-type polynomials of order α (see, [16]).

This article aims to introduce two new classes of degenerate biparametric Apostol-type polynomials. It explores various algebraic properties and relations associated with these polynomials. The derived results extend certain relations and identities of the corresponding polynomials.

2 Degenerate biparametric Apostol-type polynomials

In this section, we introduce two families of degenerate biparametric Apostol-type polynomials. We also derive certain results for these polynomials.

Definition 2.1. For arbitrary real or complex parameter α and for $a \in \mathbb{Z}^+$, the degenerate biparametric Apostol-type polynomials $\mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)$ and $\mathcal{P}_n^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)$, are defined, in a suitable neighborhood of $t = 0$, by means of the generating function:

$$\left(\frac{2^\mu t^\nu}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^\alpha (1+at)^{\frac{x}{a}} \cos\left(\frac{y}{a}\ln(1+at)\right) = \sum_{n=0}^{\infty} \mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}, \tag{8}$$

$$\left(\frac{2^\mu t^\nu}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^\alpha (1+at)^{\frac{x}{a}} \sin\left(\frac{y}{a}\ln(1+at)\right) = \sum_{n=0}^{\infty} \mathcal{P}_n^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}, \tag{9}$$

$$|t| < \left|\log\left(\frac{-1}{\lambda}\right)\right|, \quad 1^\alpha := 1.$$

Taking $x = y = 0$ in (8) we obtain the corresponding degenerate Apostol-type numbers of order α numbers defined as:

$$\left(\frac{2^\mu t^\nu}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^\alpha = \sum_{n=0}^{\infty} \mathcal{P}_n^{(\alpha)}(a;\lambda;\mu;\nu)\frac{t^n}{n!}.$$

We note the following limit case:

$$\lim_{a\to 0}\left(\frac{2^\mu t^\nu}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^\alpha (1+at)^{\frac{x}{a}} \cos\left(\frac{y}{a}\ln(1+at)\right) = \left(\frac{2^\mu t^\nu}{\lambda e^t+1}\right)^\alpha e^{xt}\cos(yt),$$

$$\lim_{a\to 0}\left(\frac{2^\mu t^\nu}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^\alpha (1+at)^{\frac{x}{a}} \sin\left(\frac{y}{a}\ln(1+at)\right) = \left(\frac{2^\mu t^\nu}{\lambda e^t+1}\right)^\alpha e^{xt}\sin(yt),$$

that is

$$\lim_{a\to 0}\sum_{n=0}^{\infty} \mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha,c)}(x,y;\lambda;\mu;\nu)\frac{t^n}{n!},$$

$$\lim_{a\to 0}\sum_{n=0}^{\infty} \mathcal{P}_n^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha,s)}(x,y;\lambda;\mu;\nu)\frac{t^n}{n!},$$

where $\mathcal{F}_n^{(\alpha,c)}(x,y;\lambda;\mu;\nu)$ and $\mathcal{F}_n^{(\alpha,s)}(x,y;\lambda;\mu;\nu)$ denotes the Apostol biparametric type polynomials. Which are introduced in this paper. Also, we note the following limits:

$$
\begin{aligned}
(-1)^{\alpha}\lim_{a\to 0}\mathcal{P}_{n}^{(\alpha,c_a)}(x,y;a;-\lambda;0;1) &= \mathcal{B}_{n,c}^{(\alpha)}(x,y,\lambda),\\
(-1)^{\alpha}\lim_{a\to 0}\mathcal{P}_{n}^{(\alpha,s_a)}(x,y;a;-\lambda;0;1) &= \mathcal{B}_{n,s}^{(\alpha)}(x,y,\lambda),\\
\lim_{a\to 0}\mathcal{P}_{n}^{(\alpha,c_a)}(x,y;a;\lambda;1;0) &= \mathcal{E}_{n,c}^{(\alpha)}(x,y,\lambda),\\
\lim_{a\to 0}\mathcal{P}_{n}^{(\alpha,s_a)}(x,y;a;\lambda;1;0) &= \mathcal{E}_{n,s}^{(\alpha)}(x,y,\lambda),\\
\lim_{a\to 0}\mathcal{P}_{n}^{(\alpha,c_a)}(x,y;a;\lambda;1;1) &= \mathcal{G}_{n,c}^{(\alpha)}(x,y,\lambda),\\
\lim_{a\to 0}\mathcal{P}_{n}^{(\alpha,s_a)}(x,y;a;\lambda;1;1) &= \mathcal{G}_{n,s}^{(\alpha)}(x,y,\lambda).
\end{aligned}
$$

Example 2.1. For $\alpha = 1, \mu = 2$, $\nu = 2, \lambda = -1$ and $a = 2$, we have

n	$\mathcal{P}_{n}^{(1,c_a)}(x,y;2;-1;2;2)$
0	-4
1	$-4x-2$
2	$-4x^2+4x+4y^2+2$
3	$-4x^3+18x^2-14x+12xy^2-18y^2-6$

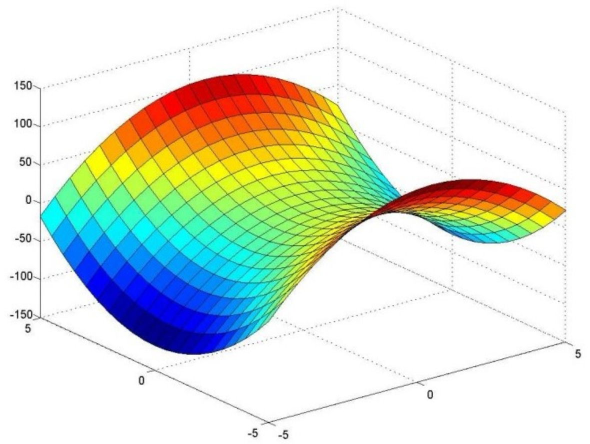

Figure 1: $-4x^2+4x+4y^2+2$

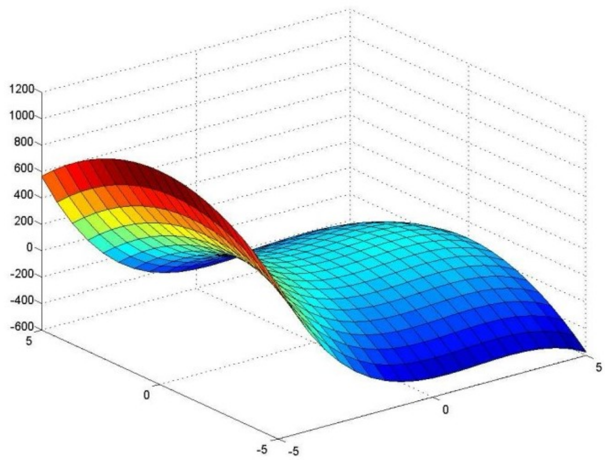

Figure 2: $-4x^3+18x^2-14x+12xy^2-18y^2-6$

Example 2.2. For $\alpha = 1, \mu = 2$, $\nu = 2, \lambda = -1$ and $a = 2$, we have

n	$\mathcal{P}_{n}^{(1,s_a)}(x,y;2;-1;2;2)$
0	$-4y$
1	$-2y-4xy$
2	$-4x^2y+12xy+\frac{4}{3}y^3-\frac{14y}{3}$
3	$-4x^3y+30x^2y+4xy^3-58xy-10y^3+20y$

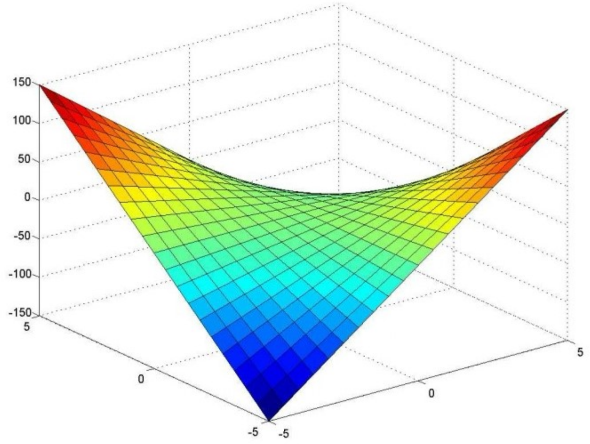

Figure 3: $-4xy-2y$

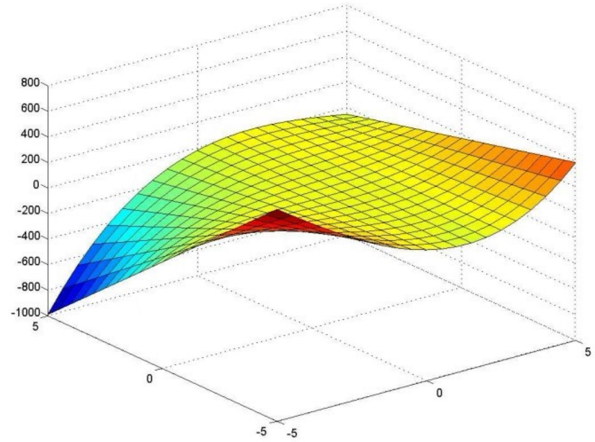

Figure 4: $-4x^2y+12xy+\frac{4}{3}y^3-\frac{14}{3}y$

3 Properties of the degenerate biparametric Apostol-type polynomials $\mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)$ and $\mathcal{P}_n^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)$

In this section, we establish some basic properties for the degenerate biparametric Apostol-type polynomials considered in the previous section.

Theorem 3.1. *The degenerate biparametric Apostol-type polynomials $\mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)$ and $\mathcal{P}_n^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)$ of order α, satisfy the following summation formula:*

$$\begin{aligned}\mathcal{P}_n^{(\alpha+\beta,c_a)}(x+z,y+w;a;\lambda;\mu;\nu) &= \sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_k^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\mathcal{P}_{n-k}^{(\beta,c_a)}(z,w;a;\lambda;\mu;\nu)\\ &\quad -\sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_k^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)\\ &\quad \times\mathcal{P}_{n-k}^{(\beta,s_a)}(z,w;a;\lambda;\mu;\nu). \end{aligned} \tag{10}$$

$$\begin{aligned}\mathcal{P}_n^{(\alpha+\beta,s_a)}(x+z,y+w;a;\lambda;\mu;\nu) &= \sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_k^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)\mathcal{P}_{n-k}^{(\beta,c_a)}(z,w;a;\lambda;\mu;\nu)\\ &\quad +\sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_k^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\\ &\quad \times\mathcal{P}_{n-k}^{(\beta,s_a)}(z,w;a;\lambda;\mu;\nu). \end{aligned} \tag{11}$$

Proof. Using (8) and the fact that, $\cos(z+w)=\cos(z)\cos(w)-\sin(z)\sin(w)$, we have.

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{P}_n^{(\alpha+\beta,c_a)}(x+z,y+w;a;\lambda;\mu;\nu)\frac{t^n}{n!} &= \left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha+\beta}(1+at)^{\frac{x+z}{a}}\cos\left(\frac{y+w}{a}\ln(1+at)\right)\\ &= \left(\sum_{n=0}^{\infty}\mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathcal{P}_n^{(\beta,c_a)}(z,w;a;\lambda;\mu;\nu)\frac{t^n}{n!}\right)\\ &\quad -\left(\sum_{n=0}^{\infty}\mathcal{P}_n^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathcal{P}_n^{(\beta,s_a)}(z,w;a;\lambda;\mu;\nu)\frac{t^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_k^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\mathcal{P}_{n-k}^{(\beta,c_a)}(z,w;a;\lambda;\mu;\nu)\frac{t^n}{n!}\\ &\quad -\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_k^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)\mathcal{P}_{n-k}^{(\beta,s_a)}(z,w;a;\lambda;\mu;\nu)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we get result (10). The proof of (11) is analogous to (10) and if we apply the known identity $\sin(z+w)=\sin(z)\cos(w)+\cos(z)\sin(w)$ the result is obtained.

□

Theorem 3.2. *For $n \in \mathbb{N}$, let $\{\mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\}_{n\geq 0}$ and $\{\mathcal{P}_n^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)\}_{n\geq 0}$ are the sequence of the degenerate biparametric Apostol-type polynomials in the variables x, y. They satisfy the following relations:*

$$(i)\ \mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu) = \mathcal{P}_n^{(\alpha,c_a)}(x+a,y;a;\lambda;\mu;\nu) - an\mathcal{P}_{n-1}^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu).$$

$$(ii)\ \mathcal{P}_n^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu) = \mathcal{P}_n^{(\alpha,s_a)}(x+a,y;a;\lambda;\mu;\nu) - an\mathcal{P}_{n-1}^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu).$$

Proof. (i). From generating function (8), we have

$$\left(\frac{2^\mu t^\nu}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^\alpha (1+at)^{\frac{x}{a}} \cos\left(\frac{y}{a}\ln(1+at)\right) = \sum_{n=0}^{\infty} \mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}.$$

$$\begin{aligned}
\left(\frac{2^\mu t^\nu}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^\alpha (1+at)^{\frac{x+a}{a}} \cos\left(\frac{y}{a}\ln(1+at)\right) &= (1+at)\sum_{n=0}^{\infty} \mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} \\
\sum_{n=0}^{\infty} \mathcal{P}_n^{(\alpha,c_a)}(x+a,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} \mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} \\
&\quad + at\sum_{n=0}^{\infty} \mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{P}_n^{(\alpha,c_a)}(x+a,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} \mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} \\
&\quad + \sum_{n=0}^{\infty} n\,\mathcal{P}_{n-1}^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\frac{at^n}{n!}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{P}_n^{(\alpha,c_a)}(x+a,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\Big[\mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu) \\
&\quad + an\mathcal{P}_{n-1}^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\Big]\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of t^n in both sides of the equation, the result is

$$\mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu) = \mathcal{P}_n^{(\alpha,c_a)}(x+a,y;a;\lambda;\mu;\nu) - an\mathcal{P}_{n-1}^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu).$$

□

The proof of **(ii)**, is analogous to **(i)** using the generating function given in (9) the result is obtained.

Theorem 3.3. *For $n \in \mathbb{N}$, let $\{\mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\}_{n\geq 0}$ and $\{\mathcal{P}_n^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)\}_{n\geq 0}$ are the sequence of degenerate biparametric Apostol–type polynomials in the variable x, y. They satisfy the following relations:*

$$(i)\ \frac{\partial \mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)}{\partial x} = \sum_{k=0}^{n-1} n(-1)^k a^k \frac{k!}{k+1}\binom{n-1}{k}\mathcal{P}_{n-1-k}^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu).$$

$$(ii)\ \frac{\partial \mathcal{P}_n^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)}{\partial x} = \sum_{k=0}^{n-1} n(-1)^k a^k \frac{k!}{k+1}\binom{n-1}{k}\mathcal{P}_{n-1-k}^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu).$$

$$(iii)\ \frac{\partial \mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)}{\partial y} = -\sum_{k=0}^{n-1} n(-1)^k a^k \frac{k!}{k+1}\binom{n-1}{k}\mathcal{P}_{n-1-k}^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu).$$

$$(iv)\ \frac{\partial \mathcal{P}_n^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)}{\partial y} = \sum_{k=0}^{n-1} n(-1)^k a^k \frac{k!}{k+1}\binom{n-1}{k}\mathcal{P}_{n-1-k}^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu).$$

Proof. (i). Partially differentiating (2.1) with respect to x, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} &= \left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}\frac{\partial}{\partial x}(1+at)^{\frac{x}{a}}\cos\left(\frac{y}{a}\ln(1+at)\right)\\
&= \left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}\cos\left(\frac{y}{a}\ln(1+at)\right)\ln(1+at)\frac{1}{a}\\
&= \left(\sum_{n=0}^{\infty}\mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}a^{n+1}t^{n+1}\frac{1}{a}\right)\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\mathcal{P}_{n-k}^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)(-1)^k a^k\binom{n}{k}\frac{k!}{k+1}\frac{t^{n+1}}{n!}.
\end{aligned}$$

Thus, we have

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\mathcal{P}_{n-1-k}^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)(-1)^k a^k n\binom{n-1}{k}\frac{k!}{k+1}\frac{t^n}{n!}.$$

Comparing the coefficients of t^n in both sides of the equation, the result is

$$\frac{\partial\mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)}{\partial x} = \sum_{k=0}^{n-1}n(-1)^k a^k\frac{k!}{k+1}\binom{n-1}{k}\mathcal{P}_{n-1-k}^{(\alpha),c_a}(x,y;a;\lambda;\mu;\nu).$$

□

The proof of **(ii)**, is analogous.

Proof. (iii). Partially differentiating (2.1) with respect to y, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{\partial}{\partial y}\mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} &= \left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}\frac{\partial}{\partial y}\cos\left(\frac{y}{a}\ln(1+at)\right)\\
&= -\left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}\sin\left(\frac{y}{a}\ln(1+at)\right)\ln(1+at)\frac{1}{a}\\
&= -\left(\sum_{n=0}^{\infty}\mathcal{P}_n^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}a^{n+1}t^{n+1}\frac{1}{a}\right)\\
&= -\sum_{n=0}^{\infty}\sum_{k=0}^{n}\mathcal{P}_{n-k}^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)(-1)^k a^k\binom{n}{k}\frac{k!}{k+1}\frac{t^{n+1}}{n!}.
\end{aligned}$$

Thus, we have

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} = -\sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\mathcal{P}_{n-1-k}^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)(-1)^k a^k n\binom{n-1}{k}\frac{k!}{k+1}\frac{t^n}{n!}.$$

Comparing the coefficients of t^n in both sides of the equation, the result is

$$\frac{\partial\mathcal{P}_n^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)}{\partial y} = -\sum_{k=0}^{n-1}n(-1)^k a^k\frac{k!}{k+1}\binom{n-1}{k}\mathcal{P}_{n-1-k}^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu).$$

□

The proof of **(iv)**, is analogous.

Theorem 3.4. *For $n\in\mathbb{N}$, let $\{\mathcal{P}_n^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)\}_{n\geq 0}$ is the sequence of degenerate biparametric Apostol-type polynomials in the variable x,y. They satisfy the following relation:*

$$\mathcal{P}_n^{(2\alpha,s_a)}(2x,2y;a;\lambda;\mu;\nu) = 2\sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_{n-k}^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\mathcal{P}_k^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu). \tag{12}$$

Proof. Consider the following expressions:

$$\left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}\cos\left(\frac{y}{a}\ln(1+at)\right)=\sum_{n=0}^{\infty}\mathcal{P}_{n}^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}, \tag{13}$$

$$\left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}\sin\left(\frac{y}{a}\ln(1+at)\right)=\sum_{n=0}^{\infty}\mathcal{P}_{n}^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}. \tag{14}$$

From (13) and (14), we have

$$\begin{aligned}\left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{2\alpha}(1+at)^{\frac{2x}{a}}\sin\left(\frac{2y}{a}\ln(1+at)\right) &= 2\sum_{n=0}^{\infty}\mathcal{P}_{n}^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}\\ &\quad\times\sum_{n=0}^{\infty}\mathcal{P}_{n}^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}\\ \sum_{n=0}^{\infty}\mathcal{P}_{n}^{(2\alpha,s_a)}(2x,2y;a;\lambda;\mu;\nu)\frac{t^n}{n!} &= 2\sum_{n=0}^{\infty}\mathcal{P}_{n}^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}\\ &\quad\times\sum_{n=0}^{\infty}\mathcal{P}_{n}^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}\\ \sum_{n=0}^{\infty}\mathcal{P}_{n}^{(2\alpha,s_a)}(2x,2y;a;\lambda;\mu;\nu)\frac{t^n}{n!} &= 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_{n-k}^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\\ &\quad\times\mathcal{P}_{k}^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}.\end{aligned}$$

□

Hence, we get assertion (12).

Theorem 3.5. *For $n\in\mathbb{N}$, let $\{\mathcal{P}_{n}^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\}_{n\geq 0}$ is the sequence of degenerate biparametric Apostol-type polynomials in the variable x,y. They satisfy the following relation:*

$$\begin{aligned}\mathcal{P}_{n}^{(\alpha,c_a)}(x,2y;a;\lambda;\mu;\nu) &= \sum_{m=0}^{n}\sum_{k=0}^{[\frac{m}{2}]}\binom{n}{m}a^{m-2k}(-1)^k y^{2k}s(m,2k)\mathcal{P}_{n-m}^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\\ &\quad-\sum_{m=0}^{n}\sum_{k=0}^{[\frac{m-1}{2}]}\binom{n}{m}a^{m-2k-1}(-1)^k y^{2k+1}s(m,2k+1)\\ &\quad\times\mathcal{P}_{n-m}^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu).\end{aligned}$$

Proof. Using the equations, (1), (2) and (8), (9) and the Cauchy series product in the resulting equation, it follows that.

For, $\phi=\cos\left(\frac{y}{a}\ln(1+at)\right)\cos\left(\frac{y}{a}\ln(1+at)\right)$

$$\begin{aligned}\left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}\phi &= \sum_{n=0}^{\infty}\mathcal{P}_{n}^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}\\ &\quad\times\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]}a^{n-2k}(-1)^k y^{2k}s(n,2k)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{[\frac{m}{2}]}\binom{n}{m}a^{m-2k}(-1)^k y^{2k}s(m,2k)\mathcal{P}_{n-m}^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\right)\frac{t^n}{n!}.\end{aligned}$$

Analogously, for $\phi=\sin\left(\frac{y}{a}\ln(1+at)\right)\sin\left(\frac{y}{a}\ln(1+at)\right)$

$$\left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}\phi=\sum_{n=0}^{\infty}\mathcal{P}_{n}^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}\sum_{n=1}^{\infty}\sum_{k=0}^{[\frac{n-1}{2}]}a^{n-2k-1}(-1)^k y^{2k+1}s(n,2k+1)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{[\frac{m-1}{2}]}\binom{n}{m}a^{m-2k-1}(-1)^k y^{2k+1}s(m,2k+1)\mathcal{P}_{n-m}^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)\right)\frac{t^n}{n!}.$$

Thus.

$$\left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}\cos^2\left(\frac{y}{a}\ln(1+at)\right)-\left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}\sin^2\left(\frac{y}{a}\ln(1+at)\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{[\frac{m}{2}]}\binom{n}{m}a^{m-2k}(-1)^k y^{2k}s(m,2k)\mathcal{P}_{n-m}^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\right)\frac{t^n}{n!}$$
$$-\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{[\frac{m-1}{2}]}\binom{n}{m}a^{m-2k-1}(-1)^k y^{2k+1}s(m,2k+1)\mathcal{P}_{n-m}^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)\right)\frac{t^n}{n!}.$$

Similarly

$$\sum_{n=0}^{\infty}\mathcal{P}_{n}^{(\alpha,c_a)}(x,2y;a;\lambda;\mu;\nu))\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{[\frac{m}{2}]}\binom{n}{m}a^{m-2k}(-1)^k y^{2k}s(m,2k)\mathcal{P}_{n-m}^{(\alpha,c_a)}(x,y;a;\lambda;\mu;\nu)\right)\frac{t^n}{n!}$$
$$-\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{[\frac{m-1}{2}]}\binom{n}{m}a^{m-2k-1}(-1)^k y^{2k+1}s(m,2k+1)\mathcal{P}_{n-m}^{(\alpha,s_a)}(x,y;a;\lambda;\mu;\nu)\right)\frac{t^n}{n!}.$$

□

Equating coefficient the result is obtained.

4 Conclusions

The article aims to introduce two novel families of degenerate biparametric Apostol-type polynomials that hold significant relevance in various fields of physics, applied mathematics, and engineering. Expressions, representations, and summations of these polynomials are derived using well-established classical special functions. The results presented in this study demonstrate the effectiveness of utilizing series rearrangement techniques in the treatment of special functions theory. We have also derived several implicit addition formulas for the Apostol-type polynomials. Based on the obtained results, it is possible to extend and establish new relationships for these newly introduced polynomial families.

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