PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Special issue dedicated to Annie Cuyt on the occasion of her 60th birthday, Volume 10 · 2017 · Pages 161–168

Selected open problems in polynomial approximation and potential theory

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Communicated by L. Bos

Abstract

Selected problems related to polynomial approximation and (pluri)potential theory are presented. They were submitted by participants of the session "Multivariate polynomial approximation and pluripotential theory" of the 4th Dolomites Workshop on Constructive Approximation and Applications.

1 Introduction

In this paper we present selected problems submitted by participants of the session "Multivariate polynomial approximation and pluripotential theory" of the 4th Dolomites Workshop on Constructive Approximation and Applications. The problems regard some approximation topics, polynomial inequalities, especially Markov-type inequalities but also Bernstein estimates for rational functions related to the Green's function. Another class of problems concerns approximation on algebraic, semialgebraic or semianalytic sets. Some questions are connected with the Hölder continuity property (HCP) of the Green's function or with the opposite property called a Łojasiewicz-Siciak inequality (ŁS). We give some motivation, a statement of the problems and partial or earlier results. At the end of every section the name of the contributing author is written in parenthesis.

2 Markov's inequality and o-minimal structures

Definition 2.1. We say that a nonempty compact set $E \subset \mathbb{C}^N$ satisfies *Markov's inequality* if there exist $\varepsilon, C > 0$ such that, for each polynomial $Q \in \mathbb{C}[Z_1, \ldots, Z_N]$ and $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N$,

$$\|D^\alpha Q\|_E \le \big(C(\deg Q)^\varepsilon\big)^{|\alpha|} \|Q\|_E, \tag{1}$$

where $D^\alpha Q := \dfrac{\partial^{|\alpha|}Q}{\partial z_1^{\alpha_1} \ldots \partial z_N^{\alpha_N}}$, $|\alpha| := \alpha_1 + \ldots + \alpha_N$, $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ and $\|\cdot\|_E$ is the supremum norm on E.

In the last three decades a good deal of attention has been given to Markov's inequality as well as its various generalizations; see for instance [3, 4, 5, 8, 24, 25, 34, 51, 52, 53, 54, 59, 62, 64, 72, 73] and the bibliography therein.

Recall that the usual techniques in the study of Markov's inequality (or similar polynomial inequalities) are mostly based on (pluri)potential theory. However, Pawłucki and Pleśniak proposed in [51] a completely different and unconventional approach. Namely, they started to explore Markov's inequality in connection with subanalytic geometry, which nowadays can be regarded as a part of the larger theory of so-called o-minimal structures.

The theory of o-minimal structures is a fairly new branch of mathematics linking model theory with geometry, "tame" topology and analysis. It comes from logic and from the theory of semialgebraic, semianalytic and subanalytic sets originating in the seminal works of Łojasiewicz (and "Łojasiewicz's group" in Kraków) [45, 46], Hironaka [38], Bierstone and Milman [18], and many other researchers.

Definition 2.2. A set $A \subset \mathbb{R}^N$ is said to be *semianalytic* if, for each point in $\mathbb{R}^N$, we can find a neighbourhood U such that $A \cap U$ is a finite union of sets of the form

$$\{x \in U : \xi(x) = 0,\ \xi_1(x) > 0, \ldots, \xi_q(x) > 0\},$$

where $\xi, \xi_1, \ldots, \xi_q$ are real analytic functions in U; see [46]. A set $A \subset \mathbb{R}^N$ is called *subanalytic* if, for each point in $\mathbb{R}^N$, there exists a neighbourhood U such that $A \cap U$ is the projection of some relatively compact semianalytic set in $\mathbb{R}^{N+N'} = \mathbb{R}^N \times \mathbb{R}^{N'}$; see [18, 29, 30, 38]. Similarly, we can define subanalytic subsets of any real analytic manifold.

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Definition 2.3. A subset of $\mathbb{R}^N$ is said to be *semialgebraic* if it is a finite union of sets of the form

$$\{x \in \mathbb{R}^N : \xi(x) = 0,\ \xi_1(x) > 0, \dots, \xi_q(x) > 0\},$$

where $\xi, \xi_1, \dots, \xi_q \in \mathbb{R}[X_1, \dots, X_N]$; see [19].

Note that all semialgebraic sets are subanalytic.

In the 1980's Lou van den Dries had noticed that many properties of semialgebraic sets could be derived from a few simple axioms defining o-minimal structures; see [32, 33]. This was the beginning of the theory of o-minimal structures which is now rapidly developing.

O-minimal structures have found many applications in analysis, differential equations, analytic geometry, potential theory or number theory. A spectacular example is Pila's unconditional proof of a special case of the André–Oort conjecture [61].

Below is the precise definition of an o-minimal structure.

Definition 2.4. An *o-minimal structure* (on the field $(\mathbb{R}, +, \cdot)$) is a sequence $\mathcal{M} = (\mathcal{M}_n)_{n\in\mathbb{N}}$ such that, for all $n, m \in \mathbb{N}$,

(i) $\mathcal{M}_n$ is a boolean algebra of subsets of $\mathbb{R}^n$;

(ii) If $A \in \mathcal{M}_n$, $B \in \mathcal{M}_m$, then $A \times B \in \mathcal{M}_{n+m}$;

(iii) If $Q \in \mathbb{R}[X_1, \dots, X_n]$, then $Q^{-1}(0) \in \mathcal{M}_n$;

(iv) If $A \in \mathcal{M}_{n+m}$, then $\pi(A) \in \mathcal{M}_n$, where $\pi : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$ denotes the projection onto the first n coordinates;

(v) The sets in $\mathcal{M}_1$ are exactly finite unions of intervals and points.

Definition 2.5. We say that a set $A \subset \mathbb{R}^n$ is *definable* (in $\mathcal{M}$) if $A \in \mathcal{M}_n$. We say that a map $f : A \longrightarrow \mathbb{R}^m$ is definable if its graph is a definable subset of $\mathbb{R}^{n+m}$.

Definition 2.6. An o-minimal structure is said to be *polynomially bounded* if, for each definable $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, there exists $k \in \mathbb{N}$ such that $\varphi(t) = O(t^k)$ as $t \to +\infty$.

Polynomial boundedness has far-reaching implications:

- Let $U \subset \mathbb{R}^n$ be an open connected set. If a $\mathcal{C}^\infty$ function $f : U \longrightarrow \mathbb{R}$ is definable in a polynomially bounded o-minimal structure and is flat at some $a \in U$, then $f \equiv 0$; see [33].
- In polynomially bounded o-minimal structures an analogue of the Łojasiewicz inequality holds; see [33].

The two basic examples of o-minimal structures are the following:

- If $\mathcal{M}_n$ consists of the semialgebraic subsets of $\mathbb{R}^n$, then $(\mathcal{M}_n)_{n\in\mathbb{N}}$ is a polynomially bounded o-minimal structure.
- If $\mathcal{M}_n$ consists of the subsets of $\mathbb{R}^n$ that are subanalytic in the projective space $\mathbb{P}^n(\mathbb{R})$, then $(\mathcal{M}_n)_{n\in\mathbb{N}}$ is a polynomially bounded o-minimal structure.

Remark 1. There exist polynomially bounded o-minimal structures in which the sets

$$E := \{(x, y) \in \mathbb{R}^2 : 0 \le x \le \epsilon,\ \theta_1 x^r \le y \le \theta_2 x^r\},$$

where $r \in (0, +\infty) \setminus \mathbb{Q}$, $\epsilon > 0$ and $\theta_2 > \theta_1 > 0$, are definable; see [33]. One can easily see that these sets are not subanalytic.

Remark 2. The set

$$E := \{(x, y) \in \mathbb{R}^2 : 0 < x \le 1,\ 0 \le y \le \exp(-x^{-1})\} \cup \{(0, 0)\}$$

is definable in any o-minimal structure which is not polynomially bounded (see [33]) and does not satisfy Markov's inequality.

For more details concerning o-minimal structures we refer the reader to [29, 32, 33].

In our opinion, it is quite important to look at Markov's inequality from the point of view of o-minimal structures. In particular, we suggest to study the following natural and highly nontrivial problem.

Problem 2.1. Given a nonempty, compact, fat (that is, $E = \overline{\mathrm{Int} E}$), and definable set $E \subset \mathbb{R}^N$, how can we decide whether E, treated as a subset of $\mathbb{C}^N$, satisfies Markov's inequality? In view of Remark 2, it seems reasonable to restrict considerations to sets definable in polynomially bounded o-minimal structures. In particular, an obvious question to ask is whether any nonempty, compact, fat and definable in a polynomially bounded o-minimal structure set $E \subset \mathbb{R}^N$ satisfies Markov's inequality.

Some results regarding this problem have been obtained by:

- Pawłucki and Pleśniak in the subanalytic context; see [51].
- Bos and Milman in the subanalytic context; see [24, 25].
- Pierzchała for certain polynomially bounded o-minimal structures; see [53, 54, 59].

In closing this section we recall a condition which is intimately linked to Markov's inequality.

Definition 2.7. We say that a nonempty compact set $E \subset \mathbb{C}^N$ has the *HCP property* if Φ_E is Hölder continuous in the following sense: there exist $\varpi, \mu > 0$ such that

$$\Phi_E(z) \le 1 + \varpi\,(\mathrm{dist}(z, E))^\mu \quad \text{for } z \in \mathbb{C}^N,\ \mathrm{dist}(z, E) \le 1\,.$$

Recall that, for a nonempty compact set $E \subset \mathbb{C}^N$, the following function

$$\Phi_E(z) := \sup\{|Q(z)|^{1/\deg Q} : Q \in \mathbb{C}[Z_1, \dots, Z_N],\ \deg Q > 0 \text{ and } \|Q\|_E \le 1\}$$

($z \in \mathbb{C}^N$) is called the *Siciak extremal function* of E; see [41, 63, 68, 69].

It is easily seen that the HCP property implies Markov's inequality. A very interesting open problem (proposed by Pleśniak) is whether the converse holds.

(R. Pierzchała)

3 The Łojasiewicz–Siciak condition

Recently there is a growing interest in the so-called Łojasiewicz–Siciak (ŁS for short) condition, which is essentially the reverse of the HCP property; see [11, 12, 15, 17, 42, 55, 56, 57, 58, 60].

Definition 3.1. We say that a nonempty compact set $E \subset \mathbb{C}^N$ satisfies the *ŁS condition* if it is polynomially convex (i.e. $E = \hat{E} := \{z \in \mathbb{C}^N : |Q(z)| \le \|Q\|_E \text{ for each } Q \in \mathbb{C}[Z_1, \ldots, Z_N]\}$) and there exist $\eta, \kappa > 0$ such that

$$\Phi_E(z) \ge 1 + \eta\,(\mathrm{dist}(z, E))^{\kappa} \quad \text{for } z \in \mathbb{C}^N,\ \mathrm{dist}(z, E) \le 1\,.$$

The interest in the ŁS condition comes from, among others, the fact that it has various interesting applications; see [11, 12, 15, 31, 56, 57]. This condition was introduced by Belghiti and Gendre around 2005; see [35]. However, as early as 1993 it was implicitly used by Siciak to prove the main result in [70].

We suggest studying the following natural problem.

Problem 3.1. Given a nonempty compact and polynomially convex set $E \subset \mathbb{C}^N$, how can we decide whether E satisfies the ŁS condition?

Below we list some results concerning the ŁS condition, which have been obtained recently.

- A family of totally disconnected, uniformly perfect planar sets satisfying the ŁS condition is constructed in [17].
- The polynomial convexity assumption in the definition of the ŁS condition posed in [17, 35] (Definition 3.1) is superfluous. See [56, Proposition 2.1].
- Each nonempty compact subset of $\mathbb{R}^N$, treated as a subset of $\mathbb{C}^N$, satisfies the ŁS condition with the exponent $\kappa = 1$. See [55, Theorem 1.1]. This result in the special case $N = 1$ was obtained independently by Bialas-Ciez and Eggink.
- Assume that a nonempty compact set $E \subset \mathbb{R}^2$ is definable in some polynomially bounded o-minimal structure. Then the following two statements are equivalent [58, Theorem 1.3]:
  1. The set E, treated as a subset of $\mathbb{C}$, satisfies the ŁS condition.
  2. The set $\mathbb{R}^2 \setminus E$ is connected and, for each $b \in \partial E := E \setminus \mathrm{Int}E$, all the interior angles of the set $\mathbb{R}^2 \setminus E$ at b are greater than 0.

  Thus, for example, the set $E_1 := \{z \in \mathbb{C} : z^n \in [0, 1]\}$ $(n \in \mathbb{N})$ satisfies the ŁS condition, whereas the set $E_2 := \{z \in \mathbb{C} : |z - 1| \le 1\} \cup \{z \in \mathbb{C} : |z + 1| \le 1\}$ does not.
- Suppose that a nonempty compact set $E \subset \mathbb{C}^N$ is polynomially convex and has the following representation:

  $$E = \{z \in \Omega : |h_1(z)| \le 1, \ldots, |h_m(z)| \le 1\},$$

  where $h_1, \ldots, h_m : \Omega \longrightarrow \mathbb{C}$ $(m \in \mathbb{N})$ are holomorphic functions and $\Omega \subset \mathbb{C}^N$ is an open neighbourhood of E. Then E satisfies the ŁS condition. See [56, Theorem 3.1].
- Assume that compact sets $E_1, \ldots, E_p \subset \mathbb{C}^N$ are definable in the same polynomially bounded o-minimal structure and satisfy the ŁS condition. Suppose moreover that $E := E_1 \cap \ldots \cap E_p \ne \emptyset$. Then E satisfies the ŁS condition. See [56, Corollary 4.2].
- [60] discusses holomorphic preimages and images of sets satisfying the ŁS condition.

(R. Pierzchała)

4 The best possible exponent in tangential Markov inequality

A tangential Markov inequality is a generalization of Markov type inequality given in Definition 2.1. We can consider the same inequality for sets $E \subset \mathbb{R}^N$ and polynomials Q with real coefficients. The set $E \subset \mathbb{K}^N$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) satisfying Markov inequality (1) is called a *Markov set*. Markov sets have to be determining for polynomials (see [62]). Recall that a set E is determining for polynomials if for every polynomial f the following implication holds: $f|_E = 0 \Rightarrow f = 0$ on $\mathbb{K}^N$. For sets, which are not determining for polynomials in $\mathbb{R}^N$, in particular for algebraic subvarieties, we can consider

Definition 4.1 (see [22])**.** A compact set $K \subset \mathbb{R}^N$ is said to admit a *tangential Markov inequality with an exponent* l if there exists a positive constant M, depending only on K, such that for all polynomials p

$$\|D_T p\|_K \le M(\deg p)^l \|p\|_K,$$

where $D_T p$ denotes any (unit) tangential derivative of p along K.

It is well known that a $\mathcal{C}^\infty$ submanifold K of $\mathbb{R}^N$ admits a tangential Markov inequality with exponent one if and only if K is algebraic (see [22]). A semialgebraic submanifold K (for the definition of a semialgebraic set see Definition 2.3) is of the $\mathcal{C}^\infty$ class, if K is $\mathbb{R}$-analytic and does not have singular points. It is shown that singular semialgebraic curve segments in $\mathbb{R}^N$ and semialgebraic surfaces in $\mathbb{R}^3$ with finitely many singular points admit a tangential Markov inequality with a finite exponent greater than 1 (see [23, 36, 43]), but some problems remain unsolved.

Problem 4.1. Does any compact subset of a semialgebraic set in $\mathbb{R}^N$ admit a tangential Markov inequality with some finite exponent?

If the answer to this question is "yes", a natural addendum is:

Problem 4.2. What is the best possible exponent in tangential Markov inequalities for these sets?

There is no known answer to the second question even for singular semialgebraic curve segments and semialgebraic surfaces with finitely many singular points. In [23] it was proved that the algebraic curve segment $\gamma_r := \{(x, x^r),\ 0 \le x \le 1\}$, where $r = q/p$ with positive integers $q \ge p$ in lowest terms, admits a tangential Markov inequality with exponent $l = 2p$ and this is the best possible exponent. For γ_r the best exponent is also equal to twice the multiplicity of the smallest complex algebraic curve containing γ_r at the singular point. Now we proceed to recall the notion of multiplicity (for more details see [28], p.102-135).

Let A be a pure p-dimensional analytic set in $\mathbb{C}^N$ and let $a \in A$. For every $(N-p)$-dimensional plane $L \ni 0$ such that a is an isolated point in $A \cap (a + L)$. Then there is a domain $U \ni a$ in $\mathbb{C}^N$ such that $A \cap U \cap (a + L) = \{a\}$ and the projection $\pi_L : A \cap U \to U'_L \subset L^\perp$ along L is a k-sheeted analytic cover, for some $k \in \mathbb{N}$. This number k is called the *multiplicity of the projection* $\pi_L|_A$ *at* a, and is denoted by $\mu_a(\pi_L|_A)$. The minimum of these multiplicities over all $(N-p)$-dimensional planes L (i.e. elements of the Grassmannian $G(N-p, N)$) is called the *multiplicity of A at the point* a. At regular points the multiplicity of an analytic set is 1. The converse is also true. Therefore, the multiplicities at singular points are important in research on the best possible exponents in tangential Markov inequality.

L. Gendre in [36] showed that the best exponent in the tangential Markov inequality at each point of a real algebraic curve A is less than or equal to twice the multiplicity of the smallest complex algebraic curve containing A. One may expect that it is always the best exponent in tangential Markov inequality for compact subsets of singular algebraic curves, but this is not true. There are sets which admit a tangential Markov inequality where the best exponent is exactly equal to the multiplicity. Examples are: an astroid, the algebraic curve given by the equation $y^2 = (1-x^2)^3$ and a family of curves $K_q = \{(t^2, t^q) : t \in [-1, 1]\}$, where $q \ge 2$ is an odd number. By Theorem 3.2 in [43], each of them admits a tangential Markov inequality with the exponent 2. Moreover, 2 is the best possible exponent for these sets and the multiplicity of the smallest complex algebraic curves containing these sets is also equal to 2 (see [44]).

Problem 4.3. Which compact subsets of singular algebraic sets do admit a tangential Markov inequality with the best exponent which is exactly equal to the multiplicity?

Problem 4.4. When is the best exponent a multiple of the multiplicity?

The conjecture is that the best exponent is a multiple of the multiplicity for a subset of a singular algebraic curve when a singular point of this curve is an endpoint of this subset.

Similar problems may be considered in L^p norms. Singular semialgebraic curve segments in $\mathbb{R}^N$ and semialgebraic surfaces in $\mathbb{R}^3$ with finitely many singular points admit a tangential Markov inequality with a finite exponent in L^p norms (see [44]), but calculation of the best exponent in this norm is a quite difficult task.

Problem 4.5. Is the best exponent in the L^p norm not less than in the uniform norm?

They are also known other characterizations in terms of Bernstein type inequalities. Baran and Pleśniak in [9] gave the characterization of semialgebraic curves and in [10] solved the problem of the characterization of semialgebraic sets of a higher dimension in the class of subanalytic sets.

Problem 4.6. Does some kind of Markov- or Bernstein-type inequality give a characterization of singular algebraic sets?

A tangential Markov inequality is not suitable for this because the analytic curve segments $(x, e^{t(x)})$, where $a \le x \le b$ and t is a fixed algebraic polynomial, admit a tangential Markov inequality with exponent 4 ([21]).

(A. Kowalska)

5 Chebyshev polynomials and polynomial inequalities

Let $q(P) = ||P||$ be a norm in the linear space $\mathbb{P}(\mathbb{C})$ of all polynomials in one variable with coefficients in $\mathbb{C}$. Let $\mathbb{P}_n(\mathbb{C}) = \{P \in \mathbb{P}(\mathbb{C}) :\ \deg P \le n\}$ and $\mathbb{M}_n(\mathbb{C})$ be the set of monic polynomials of degree n. Put

$$t_n(q) = \inf\{||P|| :\ P \in \mathbb{M}_n(\mathbb{C})\},$$

$$\mathcal{T}_n = \mathcal{T}_n(q) = \{T_n \in \mathbb{M}_n(\mathbb{C}) :\ ||T_n|| = t_n(q)\}, \quad t(q) = \inf_{n\ge1} t_n(q)^{1/n}.$$

The elements of $\mathcal{T}_n(q)$ are called *Chebyshev polynomials of degree* n for q and the quantity $t(q)$ is called the *Chebyshev constant* associated to q. A norm q has *A.Markov's property* if there exist positive constants M, m such that for each $n \ge 1$

$$||P'|| \ \le\ Mn^m ||P||, \quad P \in \mathbb{P}_n(\mathbb{C}).$$

If there exist positive constants M_1, m_1 such that

$$||P^{(k)}||/k! \ \le\ M_1^k (n^k/k!)^{m_1} ||P||, \quad P \in \mathbb{P}_n(\mathbb{C}), \quad k = 1, \ldots, n,$$

we say that the norm q has *V.Markov's property*. The *Markov exponent* and the *asymptotic exponent* of q are defined respectively by

$$m(q) = \inf\{m :\ \text{A.Markov's property holds with } m\}, \quad m^*(q) = \limsup_{k\to\infty}(m_k(q)/k)$$

where $m_k(q) = \inf\{m_k :\ ||P^{(k)}|| \le const.(\deg P)^{m_k} ||P|| \text{ for each } P\}$.

The *radial extremal function* associated to q is given by $\varphi(q,r) = \sup_{n\geq 1} \varphi_n(q,r)^{1/n}$ where

$$\varphi_n(q,r) := \sup_{|\zeta|\leq r} \sup\{||P(x+\zeta)|| : \deg P \leq n, ||P|| \leq 1\}, \quad r \geq 0.$$

The property *HCP for* q is defined by $\log \varphi(q,r) \leq Ar^{\alpha}$, $r \geq 0$ where A, α are positive constants. One can show that the two properties of q: V.Markov's property and HCP are equivalent (cf. [4]). G. Sroka in [71] proved V.Markov's property and HCP in the case of L^p norms considered on the interval $[-1,1]$. Another example of a norm q with V.Markov's property (which is easy to check) is the Wiener norm

$$||P|| = \sum_{k=0}^{\deg P} |a_k|, \quad P(\cos t) = \sum_{k=0}^{\deg P} a_k \cos(kt).$$

The above norm is very useful, it can be applied, e.g., to estimate the constant M in the inequality $||P'||_{[-1,1]} \leq M(\deg P)^2 ||P||_{[-1,1]}$ (cf. [7],[6]).

Problem 5.1. Does A.Markov's property together with $t(q) > 0$ imply V.Markov's property for any norm q?

In general (see [6]), A.Markov's property does not imply that the Chebyshev constant is strictly positive (which is necessary for V.Markov's property). On the other hand, L. Białas-Cież proved that A.Markov's property implies $t(q) > 0$ in the case of the sup norm $||P|| = ||P||_E$ on $E \subset \mathbb{C}$ (see [13]). In such a case, the above problem is reduced to one of the open problems posed by W. Pleśniak in [62]. It seems that Pleśniak's problem could be solved by considering other norms. For example, a new question is: does the norm

$$q(P) = ||P|| := \sum_{k=0}^{\infty} \left(\frac{1}{k!}\right)^m ||P^{(k)}||_{\mathbb{D}\cup\{2\}}, \ m > 1$$

(here $\mathbb{D}$ is the unit closed disc) possess V.Markov's property? In this case $t(q) \geq 1$, $m(q) \leq m$.

Problem 5.2. Does A.Markov's property with $m(q) = 1$ or $m^*(q) = 1$ imply V.Markov's property?

It is known (cf. [4]) that A.Markov's property with exponent 1 implies V.Markov's property with the same exponent. Note also that an example showing that $1 < m(q) < \infty \not\Rightarrow t(q) > 0$ is given in [6].

Problem 5.3. Does there exists for a given norm q a probability measure μ on a compact subset E of the plane $\mathbb{C}$ such that all Chebyshev polynomials for q are orthogonal with respect to μ?

Positive answers are known for $E = [a,b]$, $E = \mathbb{D}$ (unit disc) and $E = [a,b] \cup [c,d]$ with $b - a = d - c$. The last one was found by Achieser (cf. [27] for further information about other cases of sums of bounded intervals which was investigated by F. Peherstorfer and V. Totik). A positive answer implies that $\mathcal{T}_n(q)$ consists of a unique element which is well known in the case of the uniform norm (the Tonelli theorem).

Problem 5.4. Find connections between $\sup\{||P^{(k)}|| : ||P|| = 1, P \in \mathbb{P}_n(\mathbb{C})\}$ and $\max\{||T_n^{(k)}||/||T_n||, T_n \in \mathcal{T}(q)\}$. In particular, characterize norms satisfying the following inequality

$$||P^{(k)}|| \leq \max\{||T_n^{(k)}||/||T_n||, \ T_n \in \mathcal{T}(q)\}||P||, P \in \mathbb{P}_n(\mathbb{C}).$$

The famous results: V.Markov's inequality for $E = [-1,1]$ and Bernstein's inequality for the unit disc $\mathbb{D}$ give some examples for the above problem. However, it seems that the case of the union of two disjoint intervals in $\mathbb{R}$ is totally different.

(M. Baran)

6 Sharp Bernstein type inequality on the complex plane

Bernstein's inequality is well known in approximation theory and has many applications. It is stated as follows: for an algebraic polynomial P_n with degree at most n and a fixed $x \in (-1,1)$ we have

$$\left|P_n'(x)\right| \leq n \frac{1}{\sqrt{1-x^2}} ||P_n||_{[-1,1]}$$

see e.g. [20] p. 233 or [47] p. 532. This inequality is sharp in the sense that the factor $1/\sqrt{1-x^2}$, which is independent of the polynomial P_n, cannot be decreased. Although it has been generalized in various ways, see [20] or [47], it took some 80 years to determine the sharp form of Bernstein inequality for more general sets. See [2, 72] for the real case, where, in both papers, potential theory played a key role. For necessary background on potential theory, we refer to [67] and [65]. In the last ten years, asymptotically sharp Bernstein type inequalities for polynomials were established on different subsets of the complex plane, see [48, 49, 50]. In [39, 40], polynomials and different classes of rational functions are also considered on Jordan curves using, e.g., the Riemann mapping theorem in an essential way.

It is interesting to determine asymptotically sharp results on sets consisting of finitely many Jordan curves, for polynomials as well as for rational functions.

Let K be a finite union of disjoint, C^2 smooth Jordan arcs $\Gamma_1, \ldots, \Gamma_m$. Let Z be a closed set on the Riemann sphere, disjoint from K (location of possible poles). Fix $z_0 \in K$ which is not an endpoint of any Γ_k, $k = 1, 2, \ldots, m$. Then $z_0 \in \Gamma_k$ for some k, and denote the two normal vectors to Γ_k at z_0 by $\mathbf{n}_1$ and $\mathbf{n}_2$. Let D be $\mathbb{C}_\infty \setminus K$. The Green's function of D is denoted by $g_D(z,\alpha)$ where $\alpha \in D$ is the pole.

Consider a rational function f with poles in Z and of degree n (which is the maximum of the numerator and denominator degrees in simplest form).

Problem 6.1. Is it true that

$$\left|f'(z_0)\right| \le (1+o(1))\,\|f\|_K \max\left(\sum_\alpha \frac{\partial}{\partial \mathbf{n}_1} g_D(z_0,\alpha), \sum_\alpha \frac{\partial}{\partial \mathbf{n}_2} g_D(z_0,\alpha)\right) \tag{2}$$

where α runs through the poles of f counting multiplicities and $o(1)$ depends on z_0, K and Z and tends to 0 as $n \to \infty$?

Problem 6.2. Is (2) asymptotically sharp? That is, does there exist a sequence of rational functions, say f_n with $\deg f_n \to \infty$, such that the reverse inequality

$$\left|f_n'(z_0)\right| \ge (1-o(1))\,\|f_n\|_K \max\left(\sum_\alpha \frac{\partial}{\partial \mathbf{n}_1} g_D(z_0,\alpha), \sum_\alpha \frac{\partial}{\partial \mathbf{n}_2} g_D(z_0,\alpha)\right)$$

holds?

(B.Nagy)

7 Rapid approximation, Green's function and Markov inequality

For a compact set $E \subset \mathbb{C}$, let $s(E)$ be the class of continuous functions on E, which can be rapidly approximated by holomorphic polynomials:

$$s(E) \;:=\; \{f \in \mathcal{C}(E) \quad : \quad \forall\, \ell > 0 \quad \lim_{n\to\infty} n^\ell \operatorname{dist}_E(f, \mathbb{P}_n(\mathbb{C})) = 0\},$$

where $\operatorname{dist}_E(f, \mathbb{P}_n(\mathbb{C})) := \inf\{\|f - p\|_E \;:\; p \in \mathbb{P}_n(\mathbb{C})\}$ is the error of approximating the function f on the set E by polynomials of degree n or less and $\|\cdot\|_E$ is the supremum norm on E.

We denote the family of smooth functions that are $\bar\partial$-flat on E by $\mathcal{A}^\infty(E)$:

$$\mathcal{A}^\infty(E) \;:=\; \left\{\, f \in \mathcal{C}^\infty(\mathbb{C}) \;:\; \text{the function } \tfrac{\partial f}{\partial \bar z} \text{ is flat on } E \,\right\},$$

where a function $g \in \mathcal{C}^\infty(\mathbb{C})$ is said to be flat in the point z_0 if $D^\alpha g(z_0) = 0$ for all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial z^{\alpha_1} \cdot \partial \bar z^{\alpha_2}}$, $|\alpha| = \alpha_1 + \alpha_2$. This definition is slightly different than in [70], where $\mathcal{A}^\infty(E)$ stood for functions defined on E only, which will be denoted here as $\mathcal{A}^\infty(E)_{|E} := \{f_{|E} \;:\; f \in \mathcal{A}^\infty(E)\}$.

Let g_E be the Green's function of the unbounded connected component of $\mathbb{C} \setminus E$ with logarithmic pole at infinity. The set E is called *$L-$regular* if g_E is continuous. We put $E_\delta := \{z : \operatorname{dist}(z, E) \le \delta\}$.

Definition 7.1. The compact set $E \subset \mathbb{C}$ admits the *Łojasiewicz-Siciak inequality ŁS(s), where $s \ge 1$, if*

$$\exists\, M > 0 \quad \forall\, z \in E_1 \quad : \quad g_E(z) \;\ge\; M \operatorname{dist}(z, E)^s.$$

We will write that the set E admits ŁS if it admits ŁS(s) for some $s \ge 1$.

The above definition is equivalent to Defn. 3.1 in the case of $N = 1$. The Łojasiewicz-Siciak inequality has been used in order to obtain advanced approximation results, see e.g. [12, 11] (and also [70]). The interested reader is referred to [17, 55, 56] for basic information. We set out the following examples:

- if E is a compact set in $\mathbb{R}$ then E admits ŁS(1),
- the set $E := \{z \in \mathbb{C} : |z-1| \le 1 \text{ or } |z+1| \le 1\}$ does not admit ŁS(s) for any s,
- if E is the star-like set $E = E(n) := \left\{z = r \exp \frac{2\pi i j}{n} \in \mathbb{C} \;:\; 0 \le r \le 1,\; j = 1, \ldots, n\right\}$ then E admits ŁS($\frac{n}{2}$) whenever $n \in \mathbb{N} \setminus \{1\}$,
- a simply connected compact set $E \subset \mathbb{C}$ with nonempty interior, admits ŁS(s) with some $s \ge 1$ if and only if its complement to the Riemann sphere is a Hölder domain, i.e., a conformal map $\varphi : \{z \in \mathbb{C} : |z| < 1\} \to \hat{\mathbb{C}} \setminus E$ such that $\varphi(0) = \infty$ is Hölder continuous in $\{z \in \mathbb{C} : \frac{1}{2} \le |z| \le 1\}$ with exponent $1/s$ (see [70]).

The Łojasiewicz-Siciak inequality is the opposite of the Hölder Continuity Property (Defn. 2.7), which gives an upper bound of the Green's function (see e.g. [26, 1, 74, 66, 37]).

Many different versions of local Markov inequalities have been studied in the literature. Bos and Milman proposed the following

Definition 7.2. We say that E admits the *Local Markov Property* LMP if for some $m \ge 1$

$$\exists\, c, k \ge 1 \quad \forall\, z_0 \in E \quad \forall\, r \in (0,1] \quad \forall\, j, n \in \mathbb{N} \quad \forall p \in \mathbb{P}_n(\mathbb{C}) \quad |p^{(j)}(z_0)| \le \left(\tfrac{cn^k}{r^m}\right)^j \|p\|_{E \cap B(z_0, r)}.$$

Here $B(z_0, r)$ stands for the closed ball with center at z_0 and radius r. Note that LMP trivially implies the Global Markov Inequality (GMI, def. 2.1).

The connection between local and global Markov inequalities has been investigated by Bos and Milman who proved in the real case the equivalence of local and global Markov inequalities, a Sobolev type inequality and an extension property for $\mathcal{C}^\infty(E)$ functions (see [24, 25]). The proof is difficult and proceeds only in the real case making essential use of the Jackson inequality in $\mathbb{R}^N$. Unfortunately, an adaptation to the complex case of the proof given by Bos and Milman is not possible. We have constructed a counter-example in [16, Ex.3.4]. By [15, Th.1.4] and [16, Th.1.1], we have the equivalence of GMI and LMP for any polynomially convex compact set $E \subset \mathbb{C}$ admitting ŁS and HCP. However, it seems the assumption concerning HCP is too strong.

Problem 7.1 (cf. [15]). What is the weakest condition that allows to obtain an equivalence between GMI and LMP for general compact sets in the complex plane? Specifically, due to the intended application of the equivalence, it would be interesting to know whether it is sufficient to assume the L-regularity of the set instead of HCP (which of course implies L-regularity). It is still not known whether all compact subsets of the complex plane admitting GMI are L-regular. In the real case this follows from the combination of [16] and [14].

Problem 7.2 (cf. [15]). The characterization of compact sets $E \subset \mathbb{C}$, for which $\mathcal{A}^\infty(E)_{|E} \subset s(E)$, also remains an open problem, especially for totally disconnected sets. Siciak proved this property for simply connected Hölder domains, i.e. admitting ŁS [70, Th.1.10]. More recently, Belghiti, Gendre and El Ammari [11] (see also [12]), proved the same for every compact set $E \subset \mathbb{C}^N$ that admits HCP as well as ŁS.

Problem 7.3 (cf. [15]). It is of interest to look at the Wiener type characterization given by Carleson and Totik for pointwise Hölder continuity of Green's functions. Their Wiener type criterion (i.e., lower bounds for capacities) introduced in [26] implies HCP, but in order to assert the converse they needed an additional assumption, i.e., either a (geometric) cone condition or a quantitative condition (upper bounds for capacities). The examples given above suggest that both those conditions could be special cases of ŁS. It is worth investigating whether HCP in conjunction with ŁS is sufficient to assert the Wiener type criterion proposed by Carleson and Totik.

(L.Bialas-Ciez and R.Eggink)

Acknowledgements. R.Pierzchała is partially supported by the NCN grant 2015/17/B/ST1/00614. B. Nagy is supported by the Janos Bolyai Research Scholarship of the Hungarian Academy of Sciences. M.Baran, L.Bialas-Ciez and A.Kowalska are partially supported by the NCN grant No. 2013/11/B/ST1/03693.

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