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Identities for a derivation operator and their applications

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Abstract

Let $\mathcal{A}$ be a complex commutative algebra with unity $\mathbf{1}$ and let $D : \mathcal{A} \longrightarrow \mathcal{A}$ be a derivation operator (a linear operator with the property $D(ab) = bD(a) + aD(b)$). Then for arbitrary $a, b \in \mathcal{A}$ and for all positive integers k we have the following identity

$$\frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}a^j D^{(k)}(ba^{k-j}) = bD(a)^k,$$

where $D^{(k)}$ is k-th iterate of D.

In the paper we consider the algebra $\mathbb{P}(\mathbb{C}^N)$ of polynomials in N complex variables and D a derivation operator related to the A. Markov type inequality $||DP|| \le M(\deg P)^m||P||$. Using the above identity we introduce V. Markov type inequality $||D^{(k)}P|| \le A^k(\deg P)^{km}\left(\frac{1}{k!}\right)^{m-1}||P||$. We give a nontrivial example of the A. Markov inequality in the normed algebra where the V. Markov type inequality is not fulfilled. It is also shown that the Markov type condition

$$\left\|\frac{\partial}{\partial z_j}P\right\|_E \le M(\deg P)^m\|P\|_E,\ j = 1,\dots N, P \in \mathbb{P}(\mathbb{C}^N)$$

with positive constants M and m is equivalent to the following

$$\left\|\sum_{j=1}^{N}\frac{\partial^{2l}P}{\partial z_j^{2l}}\right\|_E \le M_l'(\deg P)^{2lm}\|P\|_E,\ P \in \mathbb{P}(\mathbb{C}^N)$$

with some positive constant M_l'. Here $E \subset \mathbb{R}^N$ and $l \in \mathbb{Z}_+$ is fixed.

1 Introduction

Denote by $\mathbb{P}(\mathbb{K}^N)$ the vector space of polynomials in N variables with coefficients in the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). We set $\mathbb{P}_n(\mathbb{K}^N) = \{P \in \mathbb{P}(\mathbb{K}^N) :\ \deg P \le n\}$. By $\mathbf{1}$ we mean the constant polynomial $P = 1$. Let $\mathbb{P}(j, \mathbb{K}^{N-1})$ (resp. $\mathbb{P}_n(j, \mathbb{K}^{N-1})$) be the subspace of $\mathbb{P}(\mathbb{K}^N)$ (resp. $\mathbb{P}_n(\mathbb{K}^N)$) containing only those polynomials that are independent of variable $z_j,\ j = 1, \dots, N$. In the sequel we shall consider a number of norms (and seminorms) in $\mathbb{P}(\mathbb{C}^N)$.

Let us recall that a norm (seminorm) $||\cdot||$ is *submultiplicative* if for every $P, Q \in \mathbb{P}(\mathbb{C}^N)$, $||PQ|| \le ||P|| \cdot ||Q||$ and $||\mathbf{1}|| = 1$. A norm (seminorm) ρ is *spectral* if for any $P \in \mathbb{P}(\mathbb{C}^N)$,

$$\rho(P^k) = \rho(P)^k,\ k \ge 1.$$

We shall be interested in getting lower estimates for constants M_k in the inequality of type $||P^{(k)}|| \le M_k(\deg P)^{mk}||P||$, $P \in \mathbb{P}(\mathbb{C})$ and its generalizations. It will be possible for special kinds of norms that satisfy some additional conditions.

A norm (seminorm) $||\cdot||$ is *factorizable* if there exists a submultiplicative norm (seminorm) $||\cdot||_0$ such that

$$||PQ|| \le ||P||_0||Q||,\ P, Q \in \mathbb{P}(\mathbb{K}^N).$$

The optimal $||\cdot||_0$ is given by the formula

$$||P||_0 = \sup\{||PQ|| :\ Q \in \mathbb{P}(\mathbb{C}^N),\ ||Q|| = 1\}.$$

A norm $||\cdot||$ is factorizable if and only if there exist positive constants C_l such that for any $P \in \mathbb{P}(\mathbb{K}^N)$ we have $||x_lP|| \le C_l||P||$, $l = 1, \dots, N$ (this means continuity of linear mappings $P \longrightarrow x_lP$, c.f. [7]).

Example 1.1. 1) Each submultiplicative norm (seminorm) is factorizable.

2) A supremum norm is factorizable.

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3) If $||\cdot||$ is a submultiplicative norm (seminorm) then, as a special case of known facts, $\rho(P) = \lim_{n\to\infty} ||P^n||^{1/n} = \inf_{n\geq 1} ||P^n||^{1/n}$ is a spectral seminorm (often it is a norm).
4) If E is a bounded Borel subset of $\mathbb{C}^N$, μ is a probability measure on E then for each $p \geq 1$ we have the factorizable norm $||P||_p = \left(\int_E |P|^p d\mu\right)^{1/p}$.
5) We define $||P|| = \sum_{j=0}^{\infty} \alpha_j \frac{|P^{(j)}(0)|}{j!}$ for $P \in \mathbb{P}(\mathbb{K})$, where $\alpha_j = 1$ for any even integer j and $(\alpha_{2k+1})_{k=0}^{\infty}$ is some unbounded sequence. Then for every $k \in \mathbb{Z}_+$, we have $\|x^{2k}\| = 1$ and $\|x^{2k-1}\| = \alpha_{2k-1}$. So there is no constant C such that for every $k \in \mathbb{Z}_+$, $\|x^{2k+1}\| \leq C\|x^{2k}\|$, thus this norm is not factorizable.

Let $m > 0$. A compact subset E of $\mathbb{K}^N$ is called a *Markov set* with the exponent m if for every $P \in \mathbb{P}(\mathbb{C}^N)$ the following *Markov inequality* holds:

$$\left\| \frac{\partial}{\partial x_j} P \right\|_E \leq M(\deg P)^m \|P\|_E, \text{ for } j = 1, \ldots, N, \qquad (\mathcal{M}(m))$$

where $\|f\|_E = \max\{|f(x)| : x \in E\}$ and M is independent of P. The condition $(\mathcal{M}(m))$ is equivalent to the existence of N linearly independent vectors $v_1, \ldots, v_N$ and positive constants m_j, M_j, $j = 1, \ldots, N$ such that $m = \max_{1\leq j\leq N} m_j$ and

$$\|D_{v_j} P\|_E \leq M_j (\deg P)^{m_j} \|P\|_E \quad \text{for } j = 1, \ldots, N.$$

If E is such a set, we shall write $E \in \mathcal{M}(m)$.

A Markov set fulfilling $(\mathcal{M}(m))$ will be called an *A. Markov set* or a set with the *A. Markov property*. This is to distinguish this class of sets from another subclass formed by sets satisfying the *V. Markov property*, i.e. there exist positive constants M, m such that for all $P \in \mathbb{P}_n(\mathbb{C}^N)$ we have

$$\|D^\alpha P\|_E \leq M^{|\alpha|} \left(\frac{1}{|\alpha|!}\right)^{m-1} n^{|\alpha| m} \|P\|_E$$

(in the case $N = 1$ the above condition is equivalent to the existence of a constant M_1 such that $\|P^{(k)}\|_E \leq M_1^k k! \binom{n}{k}^m \|P\|_E$.)

If $E = [-1, 1] \subset \mathbb{C}$, then the A. Markov inequality holds with $M = 1$ and $m = 2$. Moreover, if $E = \overline{\mathbb{D}}$ then the A. Markov inequality is satisfied with $m = M = 1$ and these constants are the best possible: for each n and $P = T_n$, where T_n is the n - th Chebyshev polynomial of the first kind, we have $\|T_n\|_{[-1,1]} = 1$, $T_n'(1) = n^2$ and for $P_n(z) = z^n$ we get $\|P_n\|_{\overline{\mathbb{D}}} = 1$, $\|P_n'\|_{\overline{\mathbb{D}}} = n$. Furthermore, the famous V. Markov inequality $\|P^{(k)}\|_{[-1,1]} \leq T^{(k)}_{\deg P}(1)\|P\|_{[-1,1]}$ implies the V. Markov property for the interval $[-1, 1]$. The V. Markov property for the unit disk is easily seen.

Let us remark that applying classical A. Markov inequality k times we obtain $\|P^{(k)}\|_{[-1,1]} \leq (n(n-1)\cdots(n-k+1))^2 \|P\|_{[-1,1]}$, $P \in \mathbb{P}_n(\mathbb{C})$, which is, by the V. Markov inequality, sharp. But it gives no more useful information.

The *Markov exponent* of a A. Markov set E is by definition, the best exponent in $(\mathcal{M}(s))$, i.e., $m(E) := \inf\{s > 0 : E \in \mathcal{M}(s)\}$. If E is not an A. Markov set, we put $m(E) := \infty$. Similarly we define the Markov exponent with respect to other norms. In the one-dimensional case the constants M and m are related to certain lower bounds of the logarithmic capacity of E (cf. [10],[11]).

The importance of the A. Markov property was explained by W. Pleśniak in [22] (cf. [23], see also [5]). The notion of the Markov exponent was introduced in [9] and we refer the reader to this paper for further properties of $m(E)$ (see also [4] and [19]). The importance of the V. Markov property is a consequence of the surprising fact, proved by M. Baran and L. Białas-Cież that the V. Markov property with the exponent m is equivalent to the Hölder Continuity Property in $\mathbb{C}^N$ of the Green function V_E with the exponent $\frac{1}{m}$ (see [2]).

We can also consider other norms for polynomials and consider A. Markov and V. Markov properties for these norms. In the next section we shall give a motivation for considering the V. Markov property as a minimal possible growth of the $k-th$ derivatives.

If a norm $||\cdot||$ in $\mathbb{P}(\mathbb{K}^N)$ is fixed then for a multiindex $\alpha \in \mathbb{Z}_+^N$ we define

$$\mathcal{M}_n(\alpha) = \sup\{||D^\alpha P|| : ||P|| = 1, \deg P \leq n\}$$

and if this norm possesses the A. Markov property with respect to $\alpha = e_l$ with an exponent s_l then we define the Markov factors

$$M_k(l, s_l) = \sup\{||D^{k\alpha} P||/n^{k s_l} : ||P|| = 1, \deg P \leq n, n \geq 1\}.$$

In the case $N = 1$ we shall simply write $M_k(s)$.

In the one dimensional case we can consider the Chebyshev polynomials with respect to a given norm $q = ||\cdot||$ in $\mathbb{P}(\mathbb{C})$ and the Chebyshev constant.

Definition 1.1. Let $q = ||\cdot||$ be a fixed norm (seminorm) in $\mathbb{P}(\mathbb{C})$. Define

$$t_n(q) := \inf\{||x^n + a_{n-1}x^{n-1} + \cdots + a_0|| : a_0, \ldots, a_{n-1} \in \mathbb{C}\},$$

$$t(q) := \inf_{n\geq 1} t_n(q)^{1/n}.$$

Then $t_n(q)$ is the $n-th$ *Chebyshev constant* and $t(q)$ is the *Chebyshev constant of* q. Each monic polynomial T_n such that $||T_n|| = t_n(q)$ will be called the $n-th$ *Chebyshev polynomial of* q. If P is a fixed polynomial in $\mathbb{P}(\mathbb{C})$ then we can define $t(P) = t(q_P)$, where $q_p(Q) = ||Q \circ P||$.

In particular, $t(I) = t(q)$, where $I(z) = z$. The above definitions agree with the definition given by P. Halmos for the Chebyshev constant of an element a in a complete complex normed algebra $\mathcal{A}$ (see [13]): we can consider $q(Q) = ||Q(a)||$. Then it is known (see also [13]) that $t(a) = t(\sigma(a))$, where $\sigma(a)$ is the spectrum of a. Since $\sigma(a)$ is a nonempty compact subset of $\mathbb{C}$ it is well known that $t(\sigma(a)) = C(\sigma(a)) = d(\sigma(a))$, where $C(E)$ is the logarithmic capacity and $d(E)$ is the transfinite diameter of a compact set $E \subset \mathbb{C}$.

Let us observe that $t(P) \geq t(q)^m$ if P is a monic polynomial of degree m.

Now we consider the case $N > 1$ and for $j = 1, \ldots, N$ put

$$t_n(j,q) := \inf\{||x_j^n + a_{n-1}x_j^{n-1} + \cdots + a_0|| : \ a_0, \ldots, a_{n-1} \in \mathbb{P}_{n-1}(j, \mathbb{C}^{N-1})\}.$$

A polynomial P of the form $P = x_j^n + a_{n-1}x_j^{n-1} + \cdots + a_0$ with $a_0, \ldots, a_{n-1} \in \mathbb{P}_{n-1}(j, \mathbb{C}^{N-1})$ will be called *j-monic*.

2 Identities for derivations of polynomials in complex algebras.

Let $\mathcal{A}$ be a complex commutative algebra with unity $\mathbf{1}$. Assume that a linear operator $D : \mathcal{A} \longrightarrow \mathcal{A}$ is a derivation, i.e., it satisfies $D(ab) = bD(a) + aD(b)$. This condition, known as *the Leibniz rule*, is equivalent to the equality $D(a^2) = 2aD(a)$. Denote by $D^{(k)}$ the k-th iterate of D, with $D^{(0)} = Id_{\mathcal{A}}$. A derivation D is *locally nilpotent* if for an arbitrary $a \in \mathcal{A}$ there exists $k \in \mathbb{Z}_+$ such that $D^{(k)}(a) = 0$. If D is locally nilpotent and $a \neq 0$ then we define $\deg_D a := \max\{k \in \mathbb{Z}_+ : \ D^{(k)}a \neq 0\}$.

If D is a derivation, we can easily get the well known Leibniz formula

$$D^{(k)}(ab) = \sum_{j=0}^{k} \binom{k}{j} D^{(j)}(a) D^{(k-j)}(b),$$

that is a generalization of the Leibniz rule for $k = 1$. Very recently the following generalization of Leibniz rule was discovered (see [8] for its proof)

$$\frac{1}{k!}\sum_{j=0}^{k}(-1)^j \binom{k}{j} a^j D^{(k)}(ba^{k-j}) = bD(a)^k.$$

A first version was given by Milówka [18, 19] in 2005 in the case $\mathcal{A} = \mathbb{P}(\mathbb{C})$, $D(P) = P'$, $a = P$, $b = 1$. During 7 years nobody has been interested in this deep result. In 2012 P. Ozorka found a general version (with $b = \mathbf{1}$) of the Milówka identity and M. Baran observed that the Milówka identity implies a lower estimate for the k-th derivative of polynomials considered on planar A. Markov sets. It was a new beginning of the V. Markov type property, first considered by W. Pleśniak [21].

Let us note a special case of the above generalization of the Leibniz rule. Let $DP = v_1 D_1 P + \cdots + v_N D_N P$, where $P \in \mathbb{P}(\mathbb{C}^N)$, $v_j \in \mathbb{R}$, $v_1^2 + \cdots + v_N^2 = 1$. Then we can write

$$P(x) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j \binom{k}{j} \langle x, v\rangle^j D^{(k)}(\langle x, v\rangle^{k-j} P(x)).$$

In particular,

$$P(x) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j \binom{k}{j} x_l^j \frac{\partial^k}{\partial x_l^k}(x_l^{k-j} P(x)), \ l = 1, \ldots, N. \tag{1}$$

Proposition 2.1. *Consider a fixed norm $q = ||\cdot||$ in $\mathbb{P}(\mathbb{C}^N)$ and assume that there exist $l \in \{1, \ldots, N\}$ and $m > 0$ such that for every $k \in \mathbb{N}$ there is a positive constant $M_k(l,m)$ such that $\left\|\frac{\partial^k}{\partial x_l^k}(Q)\right\| \leq M_k(l,m)(\deg Q)^{km}||Q||$ for every $Q \in \mathbb{P}(\mathbb{C}^N)$. Then for the constants $M_k(l,m)$ we have*

$$M_k(l,m) \geq ||\mathbf{1}|| k!/(k^{km} t_k(j,q)) \geq ||\mathbf{1}|| k!/(k^{km} ||x_l^k||).$$

Hence, if q is a factorizable norm with constants C_j then we have

$$M_k(l,m) \geq B_l^k \left(\frac{1}{k!}\right)^{m-1}$$

with $B_l = C_l^{-1} e^{-m}$. Thus $\inf_{k \geq 1} \left(k!^{m-1} M_k(l,m)\right)^{1/k} > 0$. Such a situation holds in the case $||Q|| = ||Q||_p = \left(\frac{1}{2}\int_{-1}^{1} |Q(t)|^p dt\right)^{1/p}$, $p \geq 1$, where is was proved by G. Sroka [25], that $\sup_{k \geq 1} (k! M_k(2))^{1/k} < \infty$ (c.f. also [16],[15],[1] for Markov's property in L^p norms).

Proof. Applying the identity (1) to $\mathbf{1}$ (or a fact that for an l-monic polynomial P_k of degree k, $||P_k^{(k)}|| = k!||1||$) get

$$||\mathbf{1}|| \leq \frac{M_k(l,m)}{k!} k^{km} t_k(l,q) \leq \frac{M_k(l,m)}{k!} k^{km} ||x_l^k||$$

and, if q is factorizable,

$$||\mathbf{1}|| \leq M_k(l,m) \frac{k^{km}}{k!} C_l^k ||\mathbf{1}||.$$

Hence $M_k(l,m) \geq \frac{k!}{k^{km}} C_l^{-k} \geq \frac{k!}{(k!e^k)^m} C_l^{-k} = \left(\frac{1}{k!}\right)^{m-1} B_l^k$. □

A similar estimate can be obtained for the operator $DP = QP'$, where $P, Q \in \mathbb{P}(\mathbb{C})$, $\deg Q = s \geq 0$ (with the leading coefficient a_s) and a given factorizable norm $q = ||\cdot||$.

Proposition 2.2. *Consider a fixed factorizable norm $||\cdot||$ on $\mathbb{P}(\mathbb{C})$ with constant C and let $Q \in \mathbb{P}(\mathbb{C})$ be a given polynomial with $\deg Q = s \geq 0$. Assume that for the operator $DP = QP'$ we have*

$$||D^{(k)}P|| \leq \widehat{M_k}(n+(k-1)s)^{km}||P||,\ P \in \mathbb{P}_n(\mathbb{C}),$$

where $\widehat{M_k}$ is a constant, $k \geq 1$, then

$$\widehat{M_k} \geq \left(\frac{1}{k!}\right)^{m-1} B^k t_{sk}(q) \geq \left(\frac{1}{k!}\right)^{m-1} (Bt(q)^s)^k,$$

where we can take

$$B = |a_s|^s C^{-1}(\max(1, ||\mathbf{1}||e^{-ms}))^{-1}(e^{m(s+1)} + e^{ms})^{-1}.$$

Proof. We can write

$$Q^k = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}x^j D^{(k)}(x^{k-j}),$$

$$||Q^k|| \leq \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}C^j||D^{(k)}(x^{k-j})||$$

$$\leq ||\mathbf{1}||\frac{\widehat{M_k}}{k!}C^k\sum_{j=0}^{k}\binom{k}{j}(k-j+(k-1)s)^{km}$$

$$\leq ||\mathbf{1}||\frac{\widehat{M_k}}{k!}C^k k^{km}\sum_{j=0}^{k}\binom{k}{j}e^{s(k-1)m}e^{-jm}$$

$$= ||\mathbf{1}||\frac{\widehat{M_k}}{k!}C^k k^{km}e^{s(k-1)m}(1+e^{-m})^k.$$

Simple calculations give the needed result. □

The next definition is related to the idea of quasianalytic functions and its presentation in Rudin's book [24].

Definition 2.1. If $||P||_0$ is a seminorm in $\mathbb{P}(\mathbb{C})$ then we put

$$||P||_r := \sum_{k=0}^{\infty}\frac{1}{k!}||D^{(k)}P||_0 r^k,\ r > 0,$$

$$||P||_{m,r} := \sum_{k=0}^{\infty}\left(\frac{1}{k!}\right)^m ||D^{(k)}P||_0 r^k,\ m, r > 0. \tag{2}$$

If $m \geq 1$ and $||\cdot||_0$ is a submultiplicative seminorm then for every $P, Q \in \mathbb{P}(\mathbb{C})$ we have (we shall apply the following inequality $\frac{1}{k!} \leq \frac{1}{j!(k-j)!}$ which is a consequence of the basic property of $\binom{k}{j}$)

$$\begin{aligned}||PQ||_{m,r} &= \sum_{k=0}^{\infty}\left(\frac{1}{k!}\right)^m\left\|\sum_{j=0}^{k}\binom{k}{j}D^{(j)}PD^{(k-j)}Q\right\|_0 r^k\\ &\leq \sum_{k=0}^{\infty}\sum_{j=0}^{k}\left(\frac{1}{j!(k-j)!}\right)^{m-1}\frac{1}{k!}\binom{k}{j}||D^{(j)}P||_0||D^{(k-j)}Q||_0 r^k\\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\left(\frac{1}{j!}\right)^m ||D^{(j)}P||_0 r^j\left(\frac{1}{(k-j)!}\right)^m ||D^{(k-j)}Q||_0 r^{k-j}\\ &= ||P||_{m,r}\cdot||Q||_{m,r}.\end{aligned}$$

If $||x||_{m,r} < \infty$ then $||P||_{m,r}$ is at least a seminorm in $\mathbb{P}(\mathbb{C})$. Such a situation holds if $DP = P'$ and $||P||_0 = \sup\{|P(t)| :\ t \in E\}$, where E is a compact subset of $\mathbb{C}$ - $||P||_{m,r}$ is a norm. A large class of other examples is determined by the following lemma.

Lemma 2.3. *Let D be a linear derivation such that $Dx = Q$ for some $Q \in \mathbb{P}(\mathbb{C})$ with $\deg Q \leq 2$ and $||\cdot||_0$ be a submultiplicative seminorm in $\mathbb{P}(\mathbb{C})$. Then $||x||_{m,r} < \infty$ for every $m > 1$ and $r > 0$, where $||\cdot||_{m,r}$ is defined by (2).*

Proof. First, note that for any linear derivation D, which satisfies the assumptions of this lemma and every $k \in \mathbb{Z}_+$ we have

$$D^{(k)}x = \sum_{l=0}^{\left[\frac{k-1}{2}\right]} \alpha_{k,l} Q^{l+1}(Q')^{k-2l-1}(Q'')^l,$$

where $[a]$ denotes the largest integer not greater than a and the constants $\alpha_{k,l}$ are defined by the following recursive relationship:

$$\alpha_{k,0} = 1 \text{ for } k \in \mathbb{Z}_+, \alpha_{k,l} = 0 \text{ for } k \in \mathbb{Z}_+ \text{ and } l > \left[\frac{k-1}{2}\right],$$

$$\alpha_{k,l} = (k-2l)\alpha_{k-1,l-1} + (l+1)\alpha_{k-1,l}.$$

By induction one can prove that for every k, l we have $|\alpha_{k,l}| \le k!$.

Put $t := \max\{\|Q\|_0, \|Q'\|_0, \|Q''\|_0\}$. We obtain that for every $k \in \mathbb{Z}_+$,

$$\|D^{(k)}x\|_0 \le \sum_{l=0}^{\left[\frac{k-1}{2}\right]} \alpha_{k,l} t^k \le kk!t^k.$$

Since $\lim_{k\to\infty} \frac{rt}{k(k+1)^{m-2}} = 0$ if $r, t > 0$ and $m > 1$, we get that $\|x\|_{m,r} < \infty$ if $r > 0$ and $m > 1$. □

Remark 1. In the case $m = 1$ we must assume $r < 1/t$ to get that $\|\cdot\|_r$ is a submultiplicative seminorm. If r is sufficiently small then, in some sense, each norm $\|\cdot\|_{m,r}$ is close to $\|\cdot\|_0$.

Proposition 2.4. *If $\|\cdot\|_0$ is a given seminorm in $\mathbb{P}(\mathbb{C})$, $DP = P'$ then for arbitrary $m, r > 0$ and for all $P \in \mathbb{P}(\mathbb{C})$ the A. Markov type inequality*

$$\|P'\|_{m,r} \le \frac{1}{r}(\deg P)^m \|P\|_{m,r}$$

holds true.

Proof. From the fact that $P^{(k)} = 0$ for $k > \deg P$, assuming $\deg P \ge 1$, we have

$$\begin{aligned}\|P'\|_{m,r} &= \sum_{k=0}^{\deg P-1} \left(\frac{1}{k!}\right)^m \|P^{(k+1)}\|_0 r^k \\ &= \frac{1}{r} \sum_{k=0}^{\deg P-1} (k+1)^m \left(\frac{1}{(k+1)!}\right)^m \|P^{(k+1)}\|_0 r^{k+1} \\ &\le \frac{1}{r}(\deg P)^m \sum_{l=1}^{\deg P} \left(\frac{1}{l!}\right)^m \|P^{(l)}\|_0 r^l \le \frac{1}{r}(\deg P)^m \|P\|_{m,r}.\end{aligned}$$

□

The derivation $DP = aP'$, where $a \in \mathbb{C}$, is the only possible locally nilpotent derivation in $\mathbb{P}(\mathbb{C})$. In $\mathbb{P}(\mathbb{C}^N)$, $N > 1$ the family of locally nilpotent derivations is much richer, we refer to [17] where there is given a criterion. Following [17] we give a few examples: $DP = D_j P$, $j = 1, \dots, N$, $DP = D_1 P + \cdots + D_N P$, $DP = D_1 P + Q(x_1) D_2 P$ and many others. For locally nilpotent derivations an analogue of Proposition 2.4 holds.

Proposition 2.5. *Let D be a locally nilpotent derivation in $\mathbb{P}(\mathbb{C}^N)$. Then $\|DP\|_{m,r} \le \left(\frac{1}{r}\right)(\deg_D P)^m \|P\|_{m,r}$.*

In the following theorem we shall see a motivation for considering the above classes of norms.

Theorem 2.6. *The A. Markov property with an exponent $m > 1$ does not imply the V. Markov property.*

Proof. Observe that the V. Markov property with constants A, s implies $\|P^{(n)}\| \le A^n n! \|P\|$ for $n = \deg P$.

Let $m > 1$ and $\|P\|_0 = |P(0)|$ and consider the norm

$$\|P\|_{m,r} := \sum_{k=0}^{\infty} \left(\frac{1}{k!}\right)^m |P^{(k)}(0)| r^k.$$

Then $(\mathbb{P}(\mathbb{C}), \|\cdot\|_{m,r})$ is a normed algebra. One can easily see that

$$\|a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0\|_{m,r} = \sum_{k=0}^{n} \left(\frac{1}{k!}\right)^{m-1} |a_k| r^k$$

and that $T_n = z^n$ is the n-th Chebyshev polynomial for the norm $\|\cdot\|_{m,r}$. We have

$$\|T_n\|_{m,r} = \left(\frac{1}{n!}\right)^{m-1} r^n, \quad \|T_n^{(n)}\|_{m,r} = n!.$$

Hence

$$\|T_n^{(n)}\|_{m,r}/\|T_n\|_{m,r} = (n!)^m r^{-n}$$

and there is no constant A such that $\|T_n^{(n)}\|_{m,r}/\|T_n\|_{m,r} \le A^n n!$.

Let us also observe that by Proposition 2.1 we have

$$M_k(s) \ge r^{-k}(k!)^m k^{-ks},$$

(here $M_k(s)$ are constants in inequalities $\|P^{(k)}\| \le M_k(s)(\deg P)^{ks}\|P\|$) which gives $m(\|\cdot\|_{m,r}) = m$.

□

Remark 2. 1) We know that the conditions $\|P^{(n)}\| \le A^n n!\|P\|$, $\|P'\| \le M(\deg P)^m\|P\|$ are necessary for the V. Markov property to hold. We can formulate the following problem: are the two conditions sufficient for the V. Markov property? Let us recall that in the case $\|P\| = \|P\|_E$, where E is a compact subset of $\mathbb{C}$, it is known that the A. Markov property implies the needed estimate for n-th derivative (see [10] and [11]).

2) We have $\|(a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0)^{(k)}\|_{m,r}$

$$\begin{aligned}
&= \frac{n!}{[(n-k)!]^m}|a_n|r^{n-k} + \frac{n-1}{[(n-k-1)!]^m}|a_{n-1}|r^{n-k-1} + \cdots + k!|a_k| \\
&= \left(\frac{n!}{(n-k)!}\right)^m r^{-k}\left[\frac{|a_n|}{(n!)^{m-1}}r^n + (n-1)!\left(\frac{n-k}{n!}\right)^m |a_{n-1}|r^{n-1}\right. \\
&\quad \left. + \cdots + k!\left(\frac{(n-k)!}{n!}\right)^m |a_k|r^k\right] \\
&\le \left(\frac{n!}{(n-k)!}\right)^m r^{-k}\|a_n z^n + a_{n-1}z^{n-1} + \ldots a_0\|_{m,r}.
\end{aligned}$$

Moreover $\|T_n^{(k)}\|_{m,r}/\|T_n\|_{m,r} = \left(\frac{n!}{(n-k)!}\right)^m r^{-k}$. Finally we get

$$\begin{aligned}
\mathcal{M}_n(k) = \sup_{\deg P \le n} \|P^{(k)}\|_{m,r}/\|P\|_{m,r} &= \left(\frac{n!}{(n-k)!}\right)^m r^{-k} \\
&= \|T_n^{(k)}\|_{m,r}/\|T_n\|_{m,r}.
\end{aligned}$$

Is a similar situation in other cases, that is does

$$\sup\{\|P^{(k)}\|/\|P\| : \ k \le \deg P \le n\} = \|T_n^{(k)}\|/\|T_n\|?$$

There is a number of deep results that gives an affirmative answer in some class of uniform norms, e.g. $\|P\| = \|P\|_E$, where $E = \mathbb{D}_r$ (the Bernstein inequality), $E = [a, b]$ (the Vladimir Markov inequality) while for $E = [-b, -a] \cup [a, b]$ the problem seems to be open.

A quite different situation is in the case $m = 1$. If $\|P'\| \le A(\deg P)\|P\|$, then $\|P^{(k)}\| \le A^k\binom{n}{k}\|P\|$, $n = \deg P$. As a special case of Proposition 2.4 we get $\|P'\|_r \le \left(\frac{1}{r}\right)\deg P\|P\|_r$.

Now we prove the following connection between the norms $\|\cdot\|_r$ and norms defined by the norm $\|\cdot\|_0$.

Proposition 2.7. *Let $\|\cdot\|_0$ be a submultiplicative norm in commutative algebra $\mathcal{A}$, fix an element $x \in \mathcal{A}$ and put (for a fixed $r > 0$)*

$$\|P\|_r = \sum_{k=0}^{\infty} \frac{1}{k!}\|P^{(k)}(x)\|_0 r^k, \ P \in \mathbb{P}(\mathbb{C}).$$

Then

$$\sup_{|\zeta| \le r} \|P(x + \zeta\mathbf{1})\|_0 \le \|P\|_r \le (\deg P + 1)\sup_{|\zeta| \le r} \|P(x + \zeta\mathbf{1})\|_0. \tag{3}$$

Proof. We shall use two facts: $P(x + \zeta 1) = \sum_{k=0}^{\infty} \frac{1}{k!}P^{(k)}(x)\zeta^k$ and $P^{(k)}(x) = k!\rho^{-k}\frac{1}{2\pi}\int_{-\pi}^{\pi} P(x + \rho e^{it}1)e^{-ikt}dt$.

The first equality gives $\sup_{|\zeta| \le r} \|P(x + \zeta\mathbf{1})\|_0 \le \|P\|_r$. From the second equality we get

$$\frac{1}{k!}\|P^{(k)}\|_0 \le \rho^{-k}\sup_{|\zeta| \le \rho} \|P(x + \zeta\mathbf{1})\|_0,$$

which permits us to write

$$\|P\|_r \le \sum_{k=0}^{\deg P}(r/\rho)^k \sup_{|\zeta| \le \rho} \|P(x + \zeta\mathbf{1})\|_0$$

and putting $\rho = r$ we obtain (3). □

Corollary 2.8. *Assume that a submultiplicative seminorm* $\|\cdot\|_0$ *is spectral* ($\|a^n\|_0 = \|a\|_0^n$, $a \in \mathcal{A}, n \in \mathbb{Z}_+$). *Then the spectral seminorm* $\rho_r(P) = \lim_{n\to\infty} \|P^n\|_r^{1/n} = \inf_{n\geq 1} \|P^n\|_r^{1/n}$ *is given by*

$$\rho_r(P) = \sup_{|\zeta|\leq r} \|P(x+\zeta\mathbf{1})\|_0.$$

Moreover, if $\mathcal{A} = \mathcal{C}(E)$, *where* $E \subset \mathbb{C}$ *is a compact set,* $x = \mathrm{Id}_E$ *then* $\rho_r(P) = \|P\|_{E_{(r)}}$ *where* $E_{(r)} = \{z \in \mathbb{C} : \ \mathrm{dist}(z,E) \leq r\}$ *is the r-th metric hull. In particular, if* $E = \{0\}$ *we get* $\rho_r(P) = \|P\|_{\overline{\mathbb{D}}_r}$.

Proposition 2.9 (C.f. [18], Thm. 3.5)**.** *If* $\|\cdot\|$ *is a spectral seminorm in* $\mathbb{P}(\mathbb{C})$ *that satisfies the following V. Markov type inequality*

$$\|P^{(k)}\| \leq A^{k+s}(n+l)^{\alpha} \frac{n^{km}}{(k!)^{m-1}} \|P\| \ \text{ for all } P \in \mathbb{P}_n(\mathbb{C}) \text{ and } k \in \mathbb{Z}_+,$$

where $s \in \mathbb{R}, M > 0, m \geq 1, l, \alpha \geq 0$ *are constants, then*

$$\|P'\| \leq A(e^m+1)n^m\|P\|. \tag{4}$$

Proof. In the proof we shall again apply the well known inequality $\frac{k^k}{k!} \leq e^k$. We have, by the Milówka identity,

$$\begin{aligned}\|P'\|^k &\leq \left(\frac{1}{k!}\right)\sum_{j=0}^{k}\binom{k}{j}\|P\|^j A^{k+s}(n(k-j)+l)^{\alpha}(n(k-j))^{km}\|P\|^{k-j}\\ &\leq A^{k+s}(nk+l)^{\alpha}e^{mk}n^{km}\sum_{j=0}^{k}\binom{k}{j}(1-j/k)^{km}\|P\|^k\\ &\leq A^{k+s}(nk+l)^{\alpha}e^{mk}n^{km}\sum_{j=0}^{k}\binom{k}{j}e^{-jm}\|P\|^k\\ &= A^{k+s}(nk+l)^{\alpha}(e^m+1)^k n^{km}\|P\|^k.\end{aligned}$$

Hence

$$\|P'\| \leq A^{1+s/k}(e^m+1)n^m(nk+l)^{\alpha/k}\|P\|.$$

Letting $k \to \infty$ we get (4), which finishes the proof. □

Now we can use Propositions 2.6 and 2.10 to observe the inequality $\|P^{(k)}\|_{\overline{\mathbb{D}}_r} \leq (n+1)r^{-k}n^k\|P\|_{\overline{\mathbb{D}}_r}$ which together with Proposition 2.12 gives a version of the Bernstein inequality.

Corollary 2.10. *If* $r > 0$ *is fixed then for all polynomial P we have*

$$\|P'\|_{\overline{\mathbb{D}}_r} \leq (e+1)r^{-1}(\deg P)\|P\|_{\overline{\mathbb{D}}_r}.$$

With the help of the Chebyshev polynomials T_n of the first kind or their derivatives we can consider the estimates for derivatives of polynomials with respect to the uniform norm on $[-1,1]$. Let $(U_j)_{j\geq 0}$ be the family of Chebyshev polynomials of the second kind that are orthogonal on $[-1,1]$ with respect to the measure $d\mu = \sqrt{1-t^2}dt$. We have $\|U_j\|_{[-1,1]} = j+1$, $U_j^{(k)} = \frac{1}{j+1}T_{j+1}^{(k+1)}$ and $\|U_j^{(k)}\|_{[-1,1]} \leq \frac{1}{2^{k-1}}\frac{(j+1)^{2k+1}}{k!}$.

We can write $P(z) = \sum_{j=0}^{n} a_j(P)U_j(z)$, where

$$a_j(P) = \frac{2}{\pi}\int_{-1}^{1} P(t)U_j(t)\sqrt{1-t^2}dt$$

with $|a_j(P)| \leq \|P\|_{[-1,1]}$ (see [14], p. 35). Hence we get

$$\begin{aligned}\|P^{(k)}\|_{[-1,1]} &\leq \sum_{j=0}^{n}|a_j(P)|\|U_j^{(k)}\|_{[-1,1]} \leq \frac{1}{k!2^{k-1}}\sum_{j=0}^{n}(j+1)^{2k+1}\|P\|_{[-1,1]}\\ &\leq \frac{4e^2}{k!2^{k-1}}n^{2+2k}\|P\|_{[-1,1]}.\end{aligned}$$

Applying now Proposition 2.12 we obtain the following version of the A. Markov inequality.

Corollary 2.11. $\|P'\|_{[-1,1]} \leq \frac{e^2+1}{2}(\deg P)^2\|P\|_{[-1,1]}$.

Remark 3. In the multivariate case we can consider the following norms

$$\|P\|_{\mathbf{m},\mathbf{r}} = \sum_{\alpha\in\mathbb{N}^N} \frac{1}{(\alpha_1!)^{m_1}} \cdots \frac{1}{(\alpha_N!)^{m_N}} \|D^\alpha P\|_0 r_1^{\alpha_1} \cdots r_N^{\alpha_N},$$

where $\mathbf{m} = (m_1, \dots, m_N),\ m_j > 0,\ \mathbf{r} = (r_1, \dots, r_N),\ r_j > 0$. We can easily get

$$\|D_j P\|_{\mathbf{m},\mathbf{r}} \le \frac{1}{r_j} (\deg_j P)^{m_j} \|P\|_{\mathbf{m},\mathbf{r}},\ j = 1, \dots, N,$$

where $\deg_j P = \deg_{D_j} P \le \deg P$. If $m_j \ge 1,\ j = 1, \dots, N$ then $\|P\|_{\mathbf{m},\mathbf{r}}$ is a submultiplicative seminorm. We can deal with the spectral radius and some other problems as in the case presented above.

3 Testing operators for the A. Markov property.

The family of operators $\mathcal{T} = \{S_j(D_1, \dots, D_N),\ j = 1, \dots s\}$, where each S_j is a homogeneous polynomial, is a *testing family for the A. Markov property* if $\|S_j(D_1, \dots, D_N)P\| \le M_j(\deg P)^{m_j}\|P\|,\ j = 1, \dots, s$ implies $\|D_j P\| \le M(\deg P)^m \|P\|,\ j = 1, \dots, N$. If $m = m_j / \deg S_j,\ j = 1, \dots s$, such a family will be called a *strong testing family*.

Proposition 3.1 ([7]). *a) Let $\mathcal{T} = \{(D_1)^{k_1}, \dots, (D_N)^{k_N}\}$, where $k_j \in \mathbb{Z}_+,\ k_j \ge 2,\ 1 \le j \le N$ is a testing family in the case of the uniform norm on a compact set E. This is a strong testing family.*

b) An example of a testing family, which consists of exactly one element is given by $\mathcal{T} = D_1 D_2 \dots D_N$. In general, it is not a strong testing family.

One can ask about the existence of a strong testing family, which consists of exactly one element. The situation is better if we consider $E \subset \mathbb{R}^N$.

Theorem 3.2. *Let E be a compact subset of $\mathbb{R}^N,\ N \ge 2$. If $k \in \mathbb{Z}_+$ then $\mathcal{T} = \{\Delta_{2k} = (D_1)^{2k} + \dots + (D_N)^{2k}\}$ is a strong testing family. In particular the Laplace operator gives a strong testing family.*

Proof. Assume that $\|(D_1)^{2k}P + \dots + (D_N)^{2k}P\|_E \le A(\deg P)^{m_1}\|P\|_E$.

First we consider polynomials with real coefficients. We can write

$$\sum_{l=1}^N (D_l P)^{2k} = \frac{1}{(2k)!} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} P^j \Delta_{2k}(P^{2k-j}).$$

By similar arguments as in the proof of Proposition 2.12 we get

$$\|D_j P\|_E \le \left\| \left(\sum_{l=1}^N (D_l P)^{2k} \right)^{\frac{1}{2k}} \right\|_E \le M(\deg P)^{\frac{m_1}{2k}} \|P\|_E,\ j = 1, \dots, N,$$

where $M = A^{\frac{1}{2k}}(1 + e^{-\frac{m_1}{2k}}((2k)^{m_1}/(2k)!)^{\frac{1}{2k}}$.

If $P = P_1 + iP_2$, where P_1 and P_2 have real coefficients, then we can consider the family of polynomials $P_\theta = \cos\theta P_1 + \sin\theta P_2,\ \theta \in [0, 2\pi]$. By the previous case we obtain $\|D_j P_\theta\|_E \le M(\deg P)^{\frac{m_1}{2k}} \|P_\theta\|_E$. Since

$$\sup_{\theta\in[0,2\pi]} |D_j P_\theta| = |D_j P|, \quad \sup_{\theta\in[0,2\pi]} |P_\theta| = |P|,$$

we have $\|D_j P\|_E \le M(\deg P)^{\frac{m_1}{2k}} \|P\|_E,\ j = 1, \dots, N$. □

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