



Sinc-collocation methods with consistent collocation points for Fredholm integral equations of the second kind

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Abstract

Sinc-collocation methods are known to be efficient for Fredholm integral equations of the second kind, even if functions in the equations have endpoint singularity. However, existing methods have the disadvantage of inconsistent collocation points. This inconsistency complicates the implementation of such methods, particularly for large-scale problems. To overcome this drawback, this study proposes another Sinc-collocation methods with consistent collocation points. The results of a theoretical error analysis show that the proposed methods have the same convergence property as existing methods. Numerical experiments suggest the superiority of the proposed methods in terms of implementation and computational cost.

Keywords: Fredholm integral equations, Sinc numerical methods, collocation methods, tanh transformation, double-exponential transformation.

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1 Introduction

This paper is concerned with Fredholm integral equations of the second kind of the following form:

$$u(t) - \int_a^b k(t,s)u(s) ds = g(t), \quad a \leq t \leq b, \quad (1.1)$$

where k and g are given continuous functions, and u is the solution to be determined. Most numerical methods provided in the literature do not perform well when the functions k and g have derivative singularity at the endpoints. To overcome the difficulty, Rashidinia and Zarebnia [11] proposed a Sinc-collocation method, which was derived without assuming differentiability at the endpoints. The results of numerical experiments indicated that their method can achieve $O(\exp(-c\sqrt{N}))$, where N denotes approximately half the number of collocation points. However, they did not prove whether their method was guaranteed to converge. Furthermore, their method required information on the boundary values of the solution $u(t)$, which are not available in practice.

To address this situation, Okayama et al. [7] reformed their method so that it could be implemented without the information of the boundary values of $u(t)$, and proved that the reformed method can attain $O(\exp(-c\sqrt{N}))$. The method is based on the Sinc approximation combined with the Single-Exponential (SE) transformation:

$$t = \psi^{\text{SE}}(x) = \frac{b-a}{2} \tanh\left(\frac{x}{2}\right) + \frac{b+a}{2}. \quad (1.2)$$

They further improved the method by replacing the SE transformation with the Double-Exponential (DE) transformation:

$$t = \psi^{\text{DE}}(x) = \frac{b-a}{2} \tanh\left(\frac{\pi}{2} \sinh x\right) + \frac{b+a}{2}, \quad (1.3)$$

and proved that their improved method can attain a higher convergence rate: $O(\exp(-cN/\log N))$.

The methods were derived as follows. Based on the Sinc approximation with auxiliary functions, set the approximate solution $u_N(t)$ as

$$u_N(t) = c_{-N-1} \frac{b-t}{b-a} + \sum_{j=-N}^N c_j \operatorname{sinc}\left(\frac{\psi^{-1}(t) - jh}{h}\right) + c_{N+1} \frac{t-a}{b-a},$$

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where $\text{sinc } x = \sin(\pi x)/(\pi x)$, and ψ is either ψ^{SE} or ψ^{DE} . Substituting u_N into the given equation (1.1) and setting $(2N + 3)$ collocation points, $t = t_i$ ($i = -N - 1, \dots, N + 1$), as

$$t_i = \begin{cases} a & (i = -N - 1), \\ \psi(ih) & (i = -N, \dots, N), \\ b & (i = N + 1), \end{cases}$$

we obtain a system of linear equations with respect to unknown coefficients c_j ($j = -N - 1, \dots, N + 1$). Those collocation points are chosen so that $u_N(t_i) = c_i$ holds for $i = -N - 1, \dots, N + 1$. However, because of the inconsistency of the collocation points at $i = \pm(N + 1)$, the methods are rather painful to implement, particularly for *systems of Fredholm integral equations*.

As a related work, Sinc-Nyström methods have been proposed [3]¹, which were derived as follows. Set the approximate solution $v_N(t)$ as

$$v_N(t) = g(t) + h \sum_{j=-N}^N k(t, \psi(jh))v_N(\psi(jh))\psi'(jh),$$

which is obtained by discretizing the integral in (1.1). Setting $(2N + 1)$ collocation points $t = \tilde{t}_i$ ($i = -N, \dots, N$) as

$$\tilde{t}_i = \psi(ih) \quad (i = -N, \dots, N),$$

we obtain a system of linear equations with respect to unknown coefficients $v_N(\psi(jh))$ ($j = -N, \dots, N$). The collocation points \tilde{t}_i are consistent for all $i = -N, \dots, N$, which is a considerable advantage in terms of practical implementation. In fact, the consistency was effectively utilized in the case of a *system of Fredholm integral equations* [10]. However, the approximate solution $v_N(t)$ includes the given functions, $g(t)$ and $k(t, s)$, which may require a high computational cost for evaluation.

To address this issue, this study derives another Sinc-collocation methods based on the concept of the Sinc-collocation methods derived for initial value problems [5]. The approximate solution $\tilde{u}_N(t)$ is given by using $\tilde{c}_j = v_N(\psi(jh))$ as

$$\tilde{u}_N(t) = \tilde{c}_{-N} \frac{b-t}{b-a} + \sum_{j=-N}^N \left(\tilde{c}_j - \tilde{c}_{-N} \frac{b-\psi(jh)}{b-a} - \tilde{c}_N \frac{\psi(jh)-a}{b-a} \right) \text{sinc} \left(\frac{\psi^{-1}(t) - jh}{h} \right) + \tilde{c}_N \frac{t-a}{b-a},$$

where the basis functions of $\tilde{u}_N(t)$ include only elementary functions. In addition, the coefficients \tilde{c}_j ($j = -N, \dots, N$) are obtained by solving the system of linear equations of the Sinc-Nyström methods, where the consistent collocation points $t = \tilde{t}_i$ are employed. Therefore, the derived methods have an advantage in terms of both implementation and computational cost.

Furthermore, by conducting a theoretical convergence analysis, this study proves that the new method with $t = \psi^{\text{SE}}(x)$ can attain $O(\exp(-c\sqrt{N}))$, and the new method with $t = \psi^{\text{DE}}(x)$ can attain $O(\exp(-cN/\log N))$. These convergence rates are the same as those of the original Sinc-collocation methods proposed by Okayama et al. [7]. However, the proposed methods may require less computation time compared to the original methods to obtain the same precision. This is due to the difference in the system of linear equations. This point is discussed in detail in Section 5.3.

The remainder of this paper is organized as follows. In Section 2, as a preliminary, basic definitions and theorems for Sinc methods are explained. Sinc methods include two important parameters, α and d , which are used to set the mesh size, h . Section 3 describes how to find those parameters, based on the result on the smoothness property of the solution of the equation (1.1). In Section 4, the original Sinc-collocation methods are described. In Section 5, the newly proposed Sinc-collocation methods are described, which are the main result of this paper. In Section 6, some numerical examples are presented. In Section 7, convergence theorems of the new Sinc-collocation methods are proved. Section 8 presents a conclusion of this paper.

2 Fundamental definitions and theorems for Sinc methods

The Sinc approximation is a function approximation formula expressed as

$$F(x) \approx \sum_{j=-N}^N F(jh)S(j, h)(x), \quad x \in \mathbb{R}, \quad (2.1)$$

where $S(j, h)(x)$ denotes the so-called ‘‘Sinc function’’ defined by

$$S(j, h)(x) = \text{sinc} \left(\frac{x - jh}{h} \right).$$

Numerical methods based on the Sinc approximation are referred to as ‘‘Sinc methods.’’ For example, a quadrature formula obtained by integrating both sides of (2.1) as

$$\int_{-\infty}^{\infty} F(x) dx \approx \sum_{j=-N}^N F(jh) \int_{-\infty}^{\infty} S(j, h)(x) dx = h \sum_{j=-N}^N F(jh) \quad (2.2)$$

is referred to as the ‘‘Sinc quadrature.’’ Comprehensive summaries of existing Sinc methods have been provided in several books and review papers [13, 14, 15, 16]. This section provides fundamental definitions and theorems for such Sinc methods as preliminaries for the remainder of this work.

¹Although the title of the paper [3] refers to Sinc-collocation methods, the work considers Sinc-Nyström methods in reality.

2.1 SE-Sinc approximation and SE-Sinc quadrature on the finite interval

To apply the Sinc approximation (2.1) to a function $f(t)$ on the finite interval (a, b) , the SE transformation (1.2) is frequently utilized as

$$f(\psi^{\text{SE}}(x)) \approx \sum_{j=-N}^N f(\psi^{\text{SE}}(jh))S(j, h)(x), \quad x \in \mathbb{R},$$

which is equivalent to

$$f(t) \approx \sum_{j=-N}^N f(\psi^{\text{SE}}(jh))S(j, h)(\{\psi^{\text{SE}}\}^{-1}(t)), \quad t \in (a, b). \quad (2.3)$$

We refer to this approximation as the “SE-Sinc approximation.” Similarly, to apply the Sinc quadrature (2.2) to the integral on the finite interval (a, b) , the SE transformation (1.2) is frequently utilized as

$$\int_a^b f(t) dt = \int_{-\infty}^{\infty} f(\psi^{\text{SE}}(x))\{\psi^{\text{SE}}\}'(x) dx \approx h \sum_{j=-N}^N f(\psi^{\text{SE}}(jh))\{\psi^{\text{SE}}\}'(jh). \quad (2.4)$$

We refer to this approximation as the “SE-Sinc quadrature.” The function space defined below is important in formalizing an error analysis of these approximations.

Definition 2.1. Let α be a positive constant, and let \mathcal{D} be a bounded and simply-connected domain (or Riemann surface) that satisfies $(a, b) \subset \mathcal{D}$. Then, $\mathbf{L}_\alpha(\mathcal{D})$ denotes the family of all functions f satisfying the following conditions: (i) f is analytic in \mathcal{D} ; (ii) there exists a constant C such that for all z in \mathcal{D} ,

$$|f(z)| \leq C|Q(z)|^\alpha, \quad (2.5)$$

where the function Q is defined by $Q(z) = (z - a)(b - z)$.

When the SE transformation is utilized, the domain \mathcal{D} in Definition 2.1 should be an eye-shaped region: $\psi^{\text{SE}}(\mathcal{D}_d) = \{z = \psi^{\text{SE}}(\zeta) : \zeta \in \mathcal{D}_d\}$ (see also an example figure [7, Fig. 1]), where \mathcal{D}_d is a strip domain defined by $\mathcal{D}_d = \{\zeta \in \mathbb{C} : |\text{Im } \zeta| < d\}$ for a positive constant d . Then, error analyses of the above approximations are described as follows.

Theorem 2.1 (Stenger [13, Theorem 4.2.5]). Let $f \in \mathbf{L}_\alpha(\psi^{\text{SE}}(\mathcal{D}_d))$ for d with $0 < d < \pi$, let N be a positive integer, and let h be selected by the formula

$$h = \sqrt{\frac{\pi d}{\alpha N}}. \quad (2.6)$$

Then, there exists a constant C that is independent of N , such that

$$\max_{a \leq t \leq b} \left| f(t) - \sum_{j=-N}^N f(\psi^{\text{SE}}(jh))S(j, h)(\{\psi^{\text{SE}}\}^{-1}(t)) \right| \leq C\sqrt{N} \exp(-\sqrt{\pi d \alpha N}).$$

Corollary 2.2 (Okayama et al. [9, Corollary 2.8]). Let $(fQ) \in \mathbf{L}_\alpha(\psi^{\text{SE}}(\mathcal{D}_d))$ for d with $0 < d < \pi$, let N be a positive integer, and let h be selected by the formula (2.6). Then, there exists a constant C that is independent of N , such that

$$\left| \int_a^b f(t) dt - h \sum_{j=-N}^N f(\psi^{\text{SE}}(jh))\{\psi^{\text{SE}}\}'(jh) \right| \leq C \exp(-\sqrt{\pi d \alpha N}).$$

Remark 1. In the case of the Sinc quadrature, the optimal choice of h is not (2.6), but $h = \sqrt{2\pi d/(\alpha N)}$. However, to implement the derived method easily, the mesh size h is chosen as (2.6) for both the SE-Sinc approximation and quadrature. The same applies to the DE-Sinc approximation and quadrature (described next).

2.2 DE-Sinc approximation and DE-Sinc quadrature on the finite interval

The key role of the SE transformation (1.2) is to map the whole real axis \mathbb{R} onto the finite interval (a, b) . Because the DE transformation (1.3) also plays the same role, the SE transformation may be replaced by the DE transformation in the above-mentioned approximations. It has recently been known that such replacement enhances the performance of Sinc methods [2, 16].

For example, by the replacement in the SE-Sinc approximation (2.3), we obtain

$$f(t) \approx \sum_{j=-N}^N f(\psi^{\text{DE}}(jh))S(j, h)(\{\psi^{\text{DE}}\}^{-1}(t)), \quad t \in (a, b).$$

We refer to this approximation as the “DE-Sinc approximation.” By the same replacement in the SE-Sinc quadrature (2.4), we obtain

$$\int_a^b f(t) dt = \int_{-\infty}^{\infty} f(\psi^{\text{DE}}(x))\{\psi^{\text{DE}}\}'(x) dx \approx h \sum_{j=-N}^N f(\psi^{\text{DE}}(jh))\{\psi^{\text{DE}}\}'(jh).$$

We refer to this approximation as the “DE-Sinc quadrature.” In these cases, the domain \mathcal{D} in Definition 2.1 should be $\psi^{\text{DE}}(\mathcal{D}_d) = \{z = \psi^{\text{DE}}(\zeta) : \zeta \in \mathcal{D}_d\}$ (see also an example figure [7, Fig. 2]), which is a Riemann surface infinitely rotating around the endpoints². Error analyses of the above approximations are described as follows.

Theorem 2.3 (Tanaka et al. [17, Theorem 3.1]). *Let $f \in \mathbf{L}_\alpha(\psi^{\text{DE}}(\mathcal{D}_d))$ for d with $0 < d < \pi/2$, let N be a positive integer, and let h be selected by the formula*

$$h = \frac{\log(2dN/\alpha)}{N}. \quad (2.7)$$

Then, there exists a constant C that is independent of N , such that

$$\max_{a \leq t \leq b} \left| f(t) - \sum_{j=-N}^N f(\psi^{\text{DE}}(jh))S(j, h)(\{\psi^{\text{DE}}\}^{-1}(t)) \right| \leq C \exp \left\{ \frac{-\pi dN}{\log(2dN/\alpha)} \right\}.$$

Corollary 2.4 (Okayama et al. [9, Corollary 2.9]). *Let $(fQ) \in \mathbf{L}_\alpha(\psi^{\text{DE}}(\mathcal{D}_d))$ for d with $0 < d < \pi/2$, let N be a positive integer, and let h be selected by the formula (2.7). Then, there exists a constant C that is independent of N , such that*

$$\left| \int_a^b f(t) dt - h \sum_{j=-N}^N f(\psi^{\text{DE}}(jh))\{\psi^{\text{DE}}\}'(jh) \right| \leq C \exp \left\{ \frac{-2\pi dN}{\log(2dN/\alpha)} \right\}.$$

After the replacement, the convergence rate is improved to $O(\exp(-cN/\log N))$, which is substantially higher than the previous rate of $O(\exp(-c\sqrt{N}))$.

Remark 2. The convergence rate $O(\exp(-cN/\log N))$ is not always attainable under the assumption in Theorem 2.1 or Corollary 2.2. We should note that $\psi^{\text{DE}}(\mathcal{D}_d)$ is a wider and more complicated domain than $\psi^{\text{SE}}(\mathcal{D}_d)$; that is, the condition imposed on f in the case of the DE transformation is more restricted than that in the case of the SE transformation. If the assumption of Theorem 2.3 or Corollary 2.4 is not fulfilled, the SE transformation may perform better than the DE transformation.

2.3 Generalized SE/DE-Sinc approximation on the finite interval

According to Theorems 2.1 and 2.3, the function f should belong to $\mathbf{L}_\alpha(\mathcal{D})$. This may seem impractical because $f(t)$ must be zero at the endpoints, $t = a$ and $t = b$, owing to the inequality (2.5). In reality, it suffices to consider the following function spaces.

Definition 2.2. Let \mathcal{D} be a bounded and simply-connected domain (or Riemann surface). Then, $\mathbf{HC}(\mathcal{D})$ denotes the family of all functions that are analytic in \mathcal{D} and continuous on $\overline{\mathcal{D}}$.

Definition 2.3. Let α be a constant with $0 < \alpha \leq 1$ and let \mathcal{D} be a bounded and simply-connected domain (or Riemann surface) that satisfies $(a, b) \subset \mathcal{D}$. Then, the space $\mathbf{M}_\alpha(\mathcal{D})$ consists of all functions f satisfying the following conditions: (i) $f \in \mathbf{HC}(\mathcal{D})$; (ii) there exists a constant C for all z in \mathcal{D} such that

$$\begin{aligned} |f(z) - f(a)| &\leq C|z - a|^\alpha, \\ |f(b) - f(z)| &\leq C|b - z|^\alpha. \end{aligned}$$

For functions $f \in \mathbf{M}_\alpha(\mathcal{D})$, using auxiliary functions

$$\omega_a(t) = \frac{b-t}{b-a}, \quad \omega_b(t) = \frac{t-a}{b-a},$$

the following approximations have been proposed:

$$\begin{aligned} f(t) \approx \mathcal{P}_N^{\text{SE}}[f](t) &= f(\psi^{\text{SE}}(-Nh))\omega_a(t) + f(\psi^{\text{SE}}(Nh))\omega_b(t) \\ &+ \sum_{j=-N}^N \{f(\psi^{\text{SE}}(jh)) - f(\psi^{\text{SE}}(-Nh))\omega_a(\psi^{\text{SE}}(jh)) - f(\psi^{\text{SE}}(Nh))\omega_b(\psi^{\text{SE}}(jh))\} S(j, h)(\{\psi^{\text{SE}}\}^{-1}(t)), \end{aligned} \quad (2.8)$$

$$\begin{aligned} f(t) \approx \mathcal{P}_N^{\text{DE}}[f](t) &= f(\psi^{\text{DE}}(-Nh))\omega_a(t) + f(\psi^{\text{DE}}(Nh))\omega_b(t) \\ &+ \sum_{j=-N}^N \{f(\psi^{\text{DE}}(jh)) - f(\psi^{\text{DE}}(-Nh))\omega_a(\psi^{\text{DE}}(jh)) - f(\psi^{\text{DE}}(Nh))\omega_b(\psi^{\text{DE}}(jh))\} S(j, h)(\{\psi^{\text{DE}}\}^{-1}(t)). \end{aligned} \quad (2.9)$$

We refer to (2.8) as the “generalized SE-Sinc approximation,” and (2.9) as the “generalized DE-Sinc approximation.” Error analyses of these approximations are described as follows. Here, $\|\cdot\|_{C([a,b])}$ denotes the usual uniform norm over $[a, b]$.

²More detailed explanation/figure can be found in Tanaka et al. [17]

Theorem 2.5 (Okayama [4, Theorem 3]). Let $f \in \mathbf{M}_\alpha(\psi^{\text{SE}}(\mathcal{D}_d))$ for d with $0 < d < \pi$, let N be a positive integer, and let h be selected by the formula (2.6). Then, there exists a constant C that is independent of N , such that

$$\|f - \mathcal{P}_N^{\text{SE}} f\|_{C([a,b])} \leq C\sqrt{N} \exp\left(-\sqrt{\pi d \alpha N}\right).$$

Theorem 2.6 (Okayama [4, Theorem 6]). Let $f \in \mathbf{M}_\alpha(\psi^{\text{DE}}(\mathcal{D}_d))$ for d with $0 < d < \pi/2$, let N be a positive integer, and let h be selected by the formula (2.7). Then, there exists a constant C that is independent of N , such that

$$\|f - \mathcal{P}_N^{\text{DE}} f\|_{C([a,b])} \leq C \exp\left\{\frac{-\pi d N}{\log(2dN/\alpha)}\right\}.$$

Remark 3. Although Okayama et al. also used the symbols $\mathcal{P}_N^{\text{SE}}$ and $\mathcal{P}_N^{\text{DE}}$ in their paper [7, (2.29) and (2.31)], their definitions differ from those in the present paper: (2.8) and (2.9). This is one of key differences between Okayama et al. and this study.

3 Smoothness property of the solution

To approximate the solution u according to Theorem 2.5 or 2.6, u must belong to $\mathbf{M}_\alpha(\psi^{\text{SE}}(\mathcal{D}_d))$ or $\mathbf{M}_\alpha(\psi^{\text{DE}}(\mathcal{D}_d))$. It should be noted that the parameters α and d , which indicate smoothness of the function, are used to select the mesh size h in (2.6) or (2.7). In practice, however, investigating u directly is not possible because u is an *unknown* function to be determined. To improve the situation, Okayama et al. [7] provided a sufficient condition for $u \in \mathbf{M}_\alpha(\mathcal{D})$ using the known functions g and k , as described below.

Suppose $k(z, \cdot) \in \mathbf{HC}(\mathcal{D})$ and $k(\cdot, w) \in \mathbf{HC}(\mathcal{D})$ for all $z, w \in \overline{\mathcal{D}}$, and let us introduce the integral operator $\mathcal{K} : \mathbf{HC}(\mathcal{D}) \rightarrow \mathbf{HC}(\mathcal{D})$ defined by

$$\mathcal{K}[f](t) = \int_a^b k(t,s)f(s) ds.$$

Note that the equation (1.1) can be rewritten as $(I - \mathcal{K})u = g$. Then, the following theorem holds for both $\mathcal{D} = \psi^{\text{SE}}(\mathcal{D}_d)$ and $\mathcal{D} = \psi^{\text{DE}}(\mathcal{D}_d)$.

Theorem 3.1 (Okayama et al. [7, Theorem 4.3]). Let $k(z, \cdot) \in \mathbf{HC}(\mathcal{D})$ and $k(\cdot, w) \in \mathbf{M}_\alpha(\mathcal{D})$ for all $z, w \in \overline{\mathcal{D}}$, and let also $g \in \mathbf{M}_\alpha(\mathcal{D})$. Furthermore, assume that the homogeneous equation $(I - \mathcal{K})f = 0$ has only the trivial solution $f \equiv 0$. Then, the equation (1.1) has a unique solution $u \in \mathbf{M}_\alpha(\mathcal{D})$.

This theorem suggests that the smoothness parameters α and d of the solution u can be found by investigating known functions g and k .

4 Original Sinc-collocation methods

In this section, Sinc-collocation methods derived by Okayama et al. [7] are described. These methods were derived in accordance with the standard collocation procedure.

4.1 Original SE-Sinc-collocation method

First, suppose that the assumptions in Theorem 3.1 are fulfilled with $\mathcal{D} = \psi^{\text{SE}}(\mathcal{D}_d)$. Then, set the approximate solution u_N^{SE} as

$$u_N^{\text{SE}}(t) = u_{-N-1}\omega_a(t) + \sum_{j=-N}^N u_j S(j, h) \{\psi^{\text{SE}}\}^{-1}(t) + u_{N+1}\omega_b(t). \quad (4.1)$$

Let us substitute u_N^{SE} into the equation (1.1), and approximate the integral operator \mathcal{K} by $\mathcal{K}_N^{\text{SE}}$:

$$\mathcal{K}_N^{\text{SE}}[f](t) = h \sum_{j=-N}^N k(t, \psi^{\text{SE}}(jh)) f(\psi^{\text{SE}}(jh)) \{\psi^{\text{SE}}\}'(jh). \quad (4.2)$$

In both (4.1) and (4.2), the mesh size h is selected by the formula (2.6). Finally, by setting the collocation points as

$$t_i^{\text{SE}} = \begin{cases} a & (i = -N - 1), \\ \psi^{\text{SE}}(ih) & (i = -N, \dots, N), \\ b & (i = N + 1), \end{cases} \quad (4.3)$$

we obtain a $(2N + 3) \times (2N + 3)$ system of linear equations:

$$\begin{aligned} & \{\omega_a(t_i^{\text{SE}}) - \mathcal{K}_N^{\text{SE}}[\omega_a](t_i^{\text{SE}})\} u_{-N-1} + \sum_{j=-N}^N \{\delta_{ij} - hk(t_i^{\text{SE}}, t_j^{\text{SE}}) \{\psi^{\text{SE}}\}'(jh)\} u_j + \{\omega_b(t_i^{\text{SE}}) - \mathcal{K}_N^{\text{SE}}[\omega_b](t_i^{\text{SE}})\} u_{N+1} \\ & = g(t_i^{\text{SE}}), \quad i = -N - 1, -N, \dots, N, N + 1. \end{aligned} \quad (4.4)$$

Let us define n by $n = 2N + 3$, and let E_n^{SE} and K_n^{SE} be $n \times n$ matrices defined by

$$E_n^{SE} = \left[\begin{array}{c|ccc|c} 1 & 0 & \cdots & 0 & 0 \\ \omega_a(t_{-N}^{SE}) & 1 & & 0 & \omega_b(t_{-N}^{SE}) \\ \vdots & & \ddots & & \vdots \\ \omega_a(t_N^{SE}) & 0 & & 1 & \omega_b(t_N^{SE}) \\ 0 & 0 & \cdots & 0 & 1 \end{array} \right],$$

$$K_n^{SE} = \left[\begin{array}{c|ccc|c} \mathcal{K}_N^{SE}[\omega_a](a) & \cdots & hk(a, t_j^{SE})\{\psi^{SE}\}'(jh) & \cdots & \mathcal{K}_N^{SE}[\omega_b](a) \\ \mathcal{K}_N^{SE}[\omega_a](t_{-N}^{SE}) & \cdots & hk(t_{-N}^{SE}, t_j^{SE})\{\psi^{SE}\}'(jh) & \cdots & \mathcal{K}_N^{SE}[\omega_b](t_{-N}^{SE}) \\ \vdots & & \vdots & & \vdots \\ \mathcal{K}_N^{SE}[\omega_a](t_N^{SE}) & \cdots & hk(t_N^{SE}, t_j^{SE})\{\psi^{SE}\}'(jh) & \cdots & \mathcal{K}_N^{SE}[\omega_b](t_N^{SE}) \\ \mathcal{K}_N^{SE}[\omega_a](b) & \cdots & hk(b, t_j^{SE})\{\psi^{SE}\}'(jh) & \cdots & \mathcal{K}_N^{SE}[\omega_b](b) \end{array} \right].$$

Furthermore, let $\mathbf{g}_n^{SE} = [g(a), g(t_{-N}^{SE}), \dots, g(t_N^{SE}), g(b)]^T$. Then, the resulting system of linear equations (4.4) can be written in the matrix-vector form:

$$(E_n^{SE} - K_n^{SE})\mathbf{u}_n = \mathbf{g}_n^{SE}, \tag{4.5}$$

where $\mathbf{u}_n = [u_{-N-1}, u_{-N}, \dots, u_N, u_{N+1}]^T$. By solving this for the coefficients \mathbf{u}_n , the approximate solution u_N^{SE} is determined by (4.1). This is the original SE-Sinc-collocation method. Its convergence was analyzed as follows.

Theorem 4.1 (Okayama et al. [7, Theorem 6.4]). *Suppose that the assumptions in Theorem 3.1 are fulfilled with $\mathcal{D} = \psi^{SE}(\mathcal{D}_d)$ for $d \in (0, \pi)$. Then, there exists a positive integer N_0 such that the equation (4.5) is uniquely solvable for all $N \geq N_0$. Furthermore, there exists a constant C independent of N such that for all $N \geq N_0$,*

$$\|u - u_N^{SE}\|_{C([a,b])} \leq C\sqrt{N} \exp(-\sqrt{\pi d a N}).$$

4.2 Original DE-Sinc-collocation method

Here, the original DE-Sinc-collocation method is described; it replaces the SE transformation in the original SE-Sinc-collocation method with the DE transformation. Suppose that the assumptions in Theorem 3.1 are fulfilled with $\mathcal{D} = \psi^{DE}(\mathcal{D}_d)$. Then, set the approximate solution u_N^{DE} as

$$u_N^{DE}(t) = u_{-N-1}\omega_a(t) + \sum_{j=-N}^N u_j S(j, h)(\{\psi^{DE}\}^{-1}(t)) + u_{N+1}\omega_b(t). \tag{4.6}$$

Let us substitute u_N^{DE} into the equation (1.1), and approximate the integral operator \mathcal{K} by \mathcal{K}_N^{DE} :

$$\mathcal{K}_N^{DE}[f](t) = h \sum_{j=-N}^N k(t, \psi^{DE}(jh))f(\psi^{DE}(jh))\{\psi^{DE}\}'(jh). \tag{4.7}$$

In both (4.6) and (4.7), the mesh size h is selected by the formula (2.7). Finally, by setting the collocation points as $t = t_i^{DE}$ defined by

$$t_i^{DE} = \begin{cases} a & (i = -N - 1), \\ \psi^{DE}(ih) & (i = -N, \dots, N), \\ b & (i = N + 1), \end{cases} \tag{4.8}$$

we obtain an $n \times n$ system of linear equations (recall $n = 2N + 3$):

$$(E_n^{DE} - K_n^{DE})\mathbf{u}_n = \mathbf{g}_n^{DE}, \tag{4.9}$$

where $\mathbf{g}_n^{DE} = [g(a), g(t_{-N}^{DE}), \dots, g(t_N^{DE}), g(b)]^T$, and E_n^{DE} and K_n^{DE} are $n \times n$ matrices defined by

$$E_n^{DE} = \left[\begin{array}{c|ccc|c} 1 & 0 & \cdots & 0 & 0 \\ \omega_a(t_{-N}^{DE}) & 1 & & 0 & \omega_b(t_{-N}^{DE}) \\ \vdots & & \ddots & & \vdots \\ \omega_a(t_N^{DE}) & 0 & & 1 & \omega_b(t_N^{DE}) \\ 0 & 0 & \cdots & 0 & 1 \end{array} \right],$$

$$K_n^{DE} = \left[\begin{array}{c|ccc|c} \mathcal{K}_N^{DE}[\omega_a](a) & \cdots & hk(a, t_j^{DE})\{\psi^{DE}\}'(jh) & \cdots & \mathcal{K}_N^{DE}[\omega_b](a) \\ \mathcal{K}_N^{DE}[\omega_a](t_{-N}^{DE}) & \cdots & hk(t_{-N}^{DE}, t_j^{DE})\{\psi^{DE}\}'(jh) & \cdots & \mathcal{K}_N^{DE}[\omega_b](t_{-N}^{DE}) \\ \vdots & & \vdots & & \vdots \\ \mathcal{K}_N^{DE}[\omega_a](t_N^{DE}) & \cdots & hk(t_N^{DE}, t_j^{DE})\{\psi^{DE}\}'(jh) & \cdots & \mathcal{K}_N^{DE}[\omega_b](t_N^{DE}) \\ \mathcal{K}_N^{DE}[\omega_a](b) & \cdots & hk(b, t_j^{DE})\{\psi^{DE}\}'(jh) & \cdots & \mathcal{K}_N^{DE}[\omega_b](b) \end{array} \right].$$

By solving this for the coefficients u_n in (4.9), the approximate solution u_N^{DE} is determined by (4.6). This is the original DE-Sinc-collocation method. Its convergence was analyzed as follows.

Theorem 4.2 (Okayama et al. [7, Theorem 8.2]). *Suppose that the assumptions in Theorem 3.1 are fulfilled with $\mathcal{D} = \psi^{\text{DE}}(\mathcal{D}_d)$ for $d \in (0, \pi/2)$. Then, there exists a positive integer N_0 such that the equation (4.9) is uniquely solvable for all $N \geq N_0$. Furthermore, there exists a constant C independent of N such that for all $N \geq N_0$,*

$$\|u - u_N^{\text{DE}}\|_{C([a,b])} \leq C \exp \left\{ \frac{-\pi d N}{\log(2\pi d/\alpha)} \right\}.$$

5 New Sinc-collocation methods with consistent collocation points

In this section, new Sinc-collocation methods are presented. These methods are derived based on the concept of Sinc-collocation methods derived for initial value problems [5], which are closely related to the Sinc-Nyström methods [3, 10]. Convergence theorems for the presented methods are stated, and proofs are provided in Section 7.

5.1 New SE-Sinc-collocation method with consistent collocation points

First, suppose that the assumptions in Theorem 3.1 are fulfilled with $\mathcal{D} = \psi^{\text{SE}}(\mathcal{D}_d)$. Then, approximating the equation $(I - \mathcal{K})u = g$ by $(I - \mathcal{K}_N^{\text{SE}})v_N^{\text{SE}} = g$ (i.e., applying Corollary 2.2 as $\mathcal{K} \approx \mathcal{K}_N^{\text{SE}}$), set the approximate solution v_N^{SE} as

$$v_N^{\text{SE}}(t) = g(t) + h \sum_{j=-N}^N k(t, \psi^{\text{SE}}(jh)) v_N^{\text{SE}}(\psi^{\text{SE}}(jh)) \{\psi^{\text{SE}}\}'(jh), \quad (5.1)$$

where the mesh size h is selected by the formula (2.6). Setting $(2N + 1)$ collocation points $t = \tilde{t}_i^{\text{SE}}$ ($i = -N, \dots, N$) as

$$\tilde{t}_i^{\text{SE}} = \psi^{\text{SE}}(ih) \quad (i = -N, \dots, N), \quad (5.2)$$

we obtain a $(2N + 1) \times (2N + 1)$ system of linear equations:

$$\sum_{j=-N}^N \left\{ \delta_{ij} - hk(\tilde{t}_i^{\text{SE}}, \tilde{t}_j^{\text{SE}}) \{\psi^{\text{SE}}\}'(jh) \right\} v_N^{\text{SE}}(\tilde{t}_j^{\text{SE}}) = g(\tilde{t}_i^{\text{SE}}), \quad i = -N, \dots, N. \quad (5.3)$$

Let us define m by $m = 2N + 1$, and let I_m be an $m \times m$ identity matrix, and let \tilde{K}_m^{SE} be an $m \times m$ matrix whose (i, j) -th element is $hk(\tilde{t}_i^{\text{SE}}, \tilde{t}_j^{\text{SE}}) \{\psi^{\text{SE}}\}'(jh)$ ($i, j = -N, \dots, N$); more precisely,

$$\tilde{K}_m^{\text{SE}} = \begin{bmatrix} hk(\tilde{t}_{-N}^{\text{SE}}, \tilde{t}_{-N}^{\text{SE}}) \{\psi^{\text{SE}}\}'(-Nh) & \cdots & hk(\tilde{t}_{-N}^{\text{SE}}, \tilde{t}_j^{\text{SE}}) \{\psi^{\text{SE}}\}'(jh) & \cdots & hk(\tilde{t}_{-N}^{\text{SE}}, \tilde{t}_N^{\text{SE}}) \{\psi^{\text{SE}}\}'(Nh) \\ \vdots & & \vdots & & \vdots \\ hk(\tilde{t}_i^{\text{SE}}, \tilde{t}_{-N}^{\text{SE}}) \{\psi^{\text{SE}}\}'(-Nh) & \cdots & hk(\tilde{t}_i^{\text{SE}}, \tilde{t}_j^{\text{SE}}) \{\psi^{\text{SE}}\}'(jh) & \cdots & hk(\tilde{t}_i^{\text{SE}}, \tilde{t}_N^{\text{SE}}) \{\psi^{\text{SE}}\}'(Nh) \\ \vdots & & \vdots & & \vdots \\ hk(\tilde{t}_N^{\text{SE}}, \tilde{t}_{-N}^{\text{SE}}) \{\psi^{\text{SE}}\}'(-Nh) & \cdots & hk(\tilde{t}_N^{\text{SE}}, \tilde{t}_j^{\text{SE}}) \{\psi^{\text{SE}}\}'(jh) & \cdots & hk(\tilde{t}_N^{\text{SE}}, \tilde{t}_N^{\text{SE}}) \{\psi^{\text{SE}}\}'(Nh) \end{bmatrix}.$$

Furthermore, let $\tilde{\mathbf{g}}_m^{\text{SE}} = [g(\tilde{t}_{-N}^{\text{SE}}), \dots, g(\tilde{t}_N^{\text{SE}})]^T$ and $\tilde{\mathbf{u}}_j = v_N^{\text{SE}}(jh)$ ($j = -N, \dots, N$). Then, the resulting system of linear equations (5.3) can be written in the matrix-vector form:

$$(I_m - \tilde{K}_m^{\text{SE}}) \tilde{\mathbf{u}}_m = \tilde{\mathbf{g}}_m^{\text{SE}}, \quad (5.4)$$

where $\tilde{\mathbf{u}}_m = [\tilde{u}_{-N}, \dots, \tilde{u}_N]^T$. By the use of the solution vector $\tilde{\mathbf{u}}_m$, let us introduce another approximate solution \tilde{u}_N^{SE} as

$$\tilde{u}_N^{\text{SE}}(t) = \tilde{u}_{-N} \omega_a(t) + \sum_{j=-N}^N \left(\tilde{u}_j - \tilde{u}_{-N} \frac{b - \psi^{\text{SE}}(jh)}{b - a} - \tilde{u}_N \frac{\psi^{\text{SE}}(jh) - a}{b - a} \right) S(j, h) (\{\psi^{\text{SE}}\}^{-1}(t)) + \tilde{u}_N \omega_b(t), \quad (5.5)$$

where the mesh size h is selected by the formula (2.6). Between $v_N^{\text{SE}}(t)$ and $\tilde{u}_N^{\text{SE}}(t)$, the relationship $\tilde{u}_N^{\text{SE}} = \mathcal{P}_N^{\text{SE}} v_N^{\text{SE}}$ holds. In summary, by solving the equation (5.4) for the coefficients $\tilde{\mathbf{u}}_m$, the approximate solution \tilde{u}_N^{SE} is determined by (5.5). This is the new SE-Sinc-collocation method. Its convergence theorem is written as follows, which is proved in Section 7.1.

Theorem 5.1. *Suppose that the assumptions in Theorem 3.1 are fulfilled with $\mathcal{D} = \psi^{\text{SE}}(\mathcal{D}_d)$ for $d \in (0, \pi)$. Then, there exists a positive integer N_0 such that the equation (5.4) is uniquely solvable for all $N \geq N_0$. Furthermore, there exists a constant C independent of N such that for all $N \geq N_0$,*

$$\|u - \tilde{u}_N^{\text{SE}}\|_{C([a,b])} \leq C \sqrt{N} \exp \left(-\sqrt{\pi d \alpha N} \right).$$

5.2 New DE-Sinc-collocation method with consistent collocation points

Here, the new DE-Sinc-collocation method is described; it replaces the SE transformation in the new SE-Sinc-collocation method with the DE transformation. Suppose that the assumptions in Theorem 3.1 are fulfilled with $\mathcal{D} = \psi^{\text{DE}}(\mathcal{D}_d)$. Then, approximating the equation $(I - \mathcal{K})u = g$ by $(I - \mathcal{K}_N^{\text{DE}})v_N^{\text{DE}} = g$ (i.e., applying Corollary 2.4 as $\mathcal{K} \approx \mathcal{K}_N^{\text{DE}}$), set the approximate solution v_N^{DE} as

$$v_N^{\text{DE}}(t) = g(t) + h \sum_{j=-N}^N k(t, \psi^{\text{DE}}(jh)) v_N^{\text{DE}}(\psi^{\text{DE}}(jh)) \{\psi^{\text{DE}}\}'(jh), \tag{5.6}$$

where the mesh size h is selected by the formula (2.7). Setting $(2N + 1)$ collocation points $t = \tilde{t}_i^{\text{DE}}$ ($i = -N, \dots, N$) as

$$\tilde{t}_i^{\text{DE}} = \psi^{\text{DE}}(ih) \quad (i = -N, \dots, N), \tag{5.7}$$

we obtain an $m \times m$ system of linear equations (recall $m = 2N + 1$):

$$(I_m - \tilde{K}_m^{\text{DE}}) \tilde{u}_m = \tilde{g}_m^{\text{DE}}, \tag{5.8}$$

where $\tilde{g}_m^{\text{DE}} = [g(\tilde{t}_{-N}^{\text{DE}}), \dots, g(\tilde{t}_N^{\text{DE}})]^T$, and \tilde{K}_m^{DE} is an $m \times m$ matrix whose (i, j) -th element is $hk(\tilde{t}_i^{\text{DE}}, \tilde{t}_j^{\text{DE}}) \{\psi^{\text{DE}}\}'(jh)$ ($i, j = -N, \dots, N$); more precisely,

$$\tilde{K}_m^{\text{DE}} = \begin{bmatrix} hk(\tilde{t}_{-N}^{\text{DE}}, \tilde{t}_{-N}^{\text{DE}}) \{\psi^{\text{DE}}\}'(-Nh) & \cdots & hk(\tilde{t}_{-N}^{\text{DE}}, \tilde{t}_j^{\text{DE}}) \{\psi^{\text{DE}}\}'(jh) & \cdots & hk(\tilde{t}_{-N}^{\text{DE}}, \tilde{t}_N^{\text{DE}}) \{\psi^{\text{DE}}\}'(Nh) \\ \vdots & & \vdots & & \vdots \\ hk(\tilde{t}_i^{\text{DE}}, \tilde{t}_{-N}^{\text{DE}}) \{\psi^{\text{DE}}\}'(-Nh) & \cdots & hk(\tilde{t}_i^{\text{DE}}, \tilde{t}_j^{\text{DE}}) \{\psi^{\text{DE}}\}'(jh) & \cdots & hk(\tilde{t}_i^{\text{DE}}, \tilde{t}_N^{\text{DE}}) \{\psi^{\text{DE}}\}'(Nh) \\ \vdots & & \vdots & & \vdots \\ hk(\tilde{t}_N^{\text{DE}}, \tilde{t}_{-N}^{\text{DE}}) \{\psi^{\text{DE}}\}'(-Nh) & \cdots & hk(\tilde{t}_N^{\text{DE}}, \tilde{t}_j^{\text{DE}}) \{\psi^{\text{DE}}\}'(jh) & \cdots & hk(\tilde{t}_N^{\text{DE}}, \tilde{t}_N^{\text{DE}}) \{\psi^{\text{DE}}\}'(Nh) \end{bmatrix}.$$

By the use of the solution vector \tilde{u}_m , let us introduce another approximate solution \tilde{u}_N^{DE} as

$$\tilde{u}_N^{\text{DE}}(t) = \tilde{u}_{-N} \omega_a(t) + \sum_{j=-N}^N \left(\tilde{u}_j - \tilde{u}_{-N} \frac{b - \psi^{\text{DE}}(jh)}{b - a} - \tilde{u}_N \frac{\psi^{\text{DE}}(jh) - a}{b - a} \right) S(j, h) (\{\psi^{\text{DE}}\}^{-1}(t)) + \tilde{u}_N \omega_b(t), \tag{5.9}$$

where the mesh size h is selected by the formula (2.7). Between $v_N^{\text{DE}}(t)$ and $\tilde{u}_N^{\text{DE}}(t)$, the relationship $\tilde{u}_N^{\text{DE}} = \mathcal{P}_N^{\text{DE}} v_N^{\text{DE}}$ holds. In summary, by solving the equation (5.8) for the coefficients \tilde{u}_m , the approximate solution \tilde{u}_N^{DE} is determined by (5.9). This is the new DE-Sinc-collocation method. Its convergence theorem is written as follows, which is proved in Section 7.2.

Theorem 5.2. *Suppose that the assumptions in Theorem 3.1 are fulfilled with $\mathcal{D} = \psi^{\text{DE}}(\mathcal{D}_d)$ for $d \in (0, \pi/2)$. Then, there exists a positive integer N_0 such that the equation (5.8) is uniquely solvable for all $N \geq N_0$. Furthermore, there exists a constant C independent of N such that for all $N \geq N_0$,*

$$\|u - \tilde{u}_N^{\text{DE}}\|_{C([a,b])} \leq C \exp \left\{ \frac{-\pi d N}{\log(2\pi d/\alpha)} \right\}.$$

5.3 Discussion on the differences between the original and new Sinc-collocation methods

The primary difference between the original and new Sinc-collocation methods lies in the system of linear equations to be solved. Recall that the system of equations in the original methods is (4.5) or (4.9), whereas that in the new methods is (5.4) or (5.8). In the case of the original methods, users must handle the exceptional cases at $i, j = \pm(N + 1)$, due to the inconsistency of the collocation points (4.3) or (4.8). In the case of the new methods, the systems of linear equations are considerably simple and easy to implement, owing to the consistent collocation points (5.2) or (5.7). Especially in the matrices K_n^{SE} and K_n^{DE} of the original methods, the elements in $j = \pm(N + 1)$ include a summation term $\mathcal{K}_N^{\text{SE}}$ or $\mathcal{K}_N^{\text{DE}}$, which may incur a high computational cost. In contrast, such a term is not included in the matrices \tilde{K}_m^{SE} and \tilde{K}_m^{DE} of the new methods. For this reason, the difference in the computation time between the original and new methods is significant in the case where k is computationally complex for evaluation.

One minor difference is that the size of the systems (5.4) and (5.8) is $m = 2N + 1$, which is slightly smaller than the size of the systems (4.5) and (4.9): $n = 2N + 3$. It also should be noted that the original and new methods share exactly the same basis functions. Accordingly, the number of the basis functions is the same: $n = 2N + 3$. In terms of the convergence rates, as Theorems 4.1–4.2 and Theorems 5.1–5.2 show, there is no difference between the original and new methods.

In short, the new methods have an advantage in terms of practicality of implementation and computational cost, without the loss of convergence performance. This is also confirmed in the next section.

Table 1: Values of the parameters α and d . $\pi_m = 3.14$.

	Example 6.1	Example 6.2	Example 6.3	Example 6.4
Original SE, α	1.0	0.5	0.5	1.0
New SE, α	1.0	0.5	0.5	1.0
Original DE, α	1.0	0.5	0.5	1.0
New DE, α	1.0	0.5	0.5	1.0
Original SE, d	$\pi_m/2$	π_m	π_m	$\pi_m/2$
New SE, d	$\pi_m/2$	π_m	π_m	$\pi_m/2$
Original DE, d	$\pi_m/6$	$\pi_m/2$	$\pi_m/2$	0.125
New DE, d	$\pi_m/6$	$\pi_m/2$	$\pi_m/2$	0.125

6 Numerical examples

In this section, numerical comparisons between the four methods (original SE/DE-Sinc-collocation methods and new SE/DE-Sinc-collocation methods) are presented. The computation was performed on MacBook Air with 1.7 GHz Intel Core i7 with 8GB memory, running Mac OS X 10.12.6. The computation programs were implemented in the C programming language with double-precision floating-point arithmetic, and compiled with Apple LLVM version 9.0.0 with the “-O2” option. LAPACK in Apple’s Accelerate framework was used for computation of the system of linear equations. The source code for all programs is available at <https://github.com/okayamat/sinc-colloc-fredholm>.

We consider the following four examples.

Example 6.1 (Delves and Mohamed [1, special case of Family 4 in Chap. 10]). Consider the following equation

$$u(t) - \int_0^1 tsu(s) ds = \frac{r}{(t-1/2)^2 + r^2} - t \arctan\left(\frac{1}{2r}\right), \quad 0 \leq t \leq 1,$$

whose solution is $u(t) = r/\{(t-1/2)^2 + r^2\}$. For the experiment, r was set as $r = 1/2$.

Example 6.2 (Delves and Mohamed [1, Example 4.2.5b]). Consider the following equation

$$u(t) - \int_0^{\pi/2} (ts)^{3/4} u(s) ds = t^{1/2} \left\{ 1 - \frac{\pi^2}{9} \left(\frac{\pi t}{2}\right)^{1/4} \right\}, \quad 0 \leq t \leq \pi/2,$$

whose solution is $u(t) = t^{1/2}$.

Example 6.3 (Okayama et al. [7, Example 9.4]). Consider the following equation

$$u(t) - \int_0^1 t^{\sqrt{3}-1} \sum_{l=1}^{100} s^{a_l} (1-s)^{1-b_l} u(s) ds = t^{1/2} - t^{\sqrt{3}-1} \sum_{l=1}^{100} B(a_l + 3/2, 2 - b_l), \quad 0 \leq t \leq 1,$$

whose solution is $u(t) = t^{1/2}$. Here, $B(p, q)$ is the beta function, $a_l = (3/\pi)^l$, and $b_l = (2\sqrt{2}/3)^l$.

Example 6.4 (Okayama et al. [7, Example 9.5]). Consider the following equation

$$\begin{aligned} u(t) - 2 \int_{-1}^1 \frac{(1-s^2)^{(2-t^2)/(2+t^2)}}{(2+t^2)(1+s^{20})} \{5(2+t^2)s^{18}(1-s^2) + (s^{20}+1)(s^{21}+s+2)\} u(s) ds \\ = \frac{2t}{1+t^{20}} - \frac{4}{2+t^2} B(3/2, 4/(2+t^2)), \quad -1 \leq t \leq 1, \end{aligned}$$

whose solution is $u(t) = 2t/(1+t^{20})$.

Following Okayama et al. [7], the values of the parameters α and d were chosen as shown in Table 1. Here, $\pi_m = 3.14$ was used as a slightly smaller value than π . Because there is no difference between the assumptions in the theorems of the original and new methods, the methods share the same values for those parameters. The number of lines of the main programs is shown in Table 2. The table reveals the advantage of the new SE/DE-Sinc collocation methods in terms of implementation. Furthermore, note that the differences between the four examples are small, even though Examples 6.3 and 6.4 are extremely difficult for Gaussian quadrature-based methods.

Figures 1–4 show the convergence profile with respect to N . We observe that the original and new SE-Sinc-collocation methods converge at the same rate, $O(\exp(-c\sqrt{N}))$, which is suggested by Theorems 4.1 and 5.1. We also observe that the original and new DE-Sinc-collocation methods converge at the same rate, $O(\exp(-cN/\log N))$, which is suggested by Theorems 4.2 and 5.2. However, especially in Example 6.3, the notable advantage of the new SE/DE-Sinc-collocation methods may be observed in terms of the computational cost. The convergence profile with respect to the computation time is displayed in Figure 5. The difference in the performance is due to the functions k and g in Example 6.3, which are relatively computationally complex for evaluation. In other examples, such a significant difference was not observed.

Table 2: Number of lines of the main programs.

	Example 6.1	Example 6.2	Example 6.3	Example 6.4
Original SE	171	169	204	182
Original DE	171	169	204	183
New SE	127	125	160	138
New DE	127	125	160	139

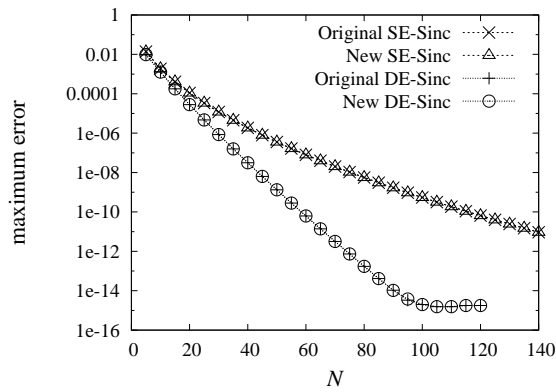


Figure 1: Convergence with respect to N for Example 6.1.

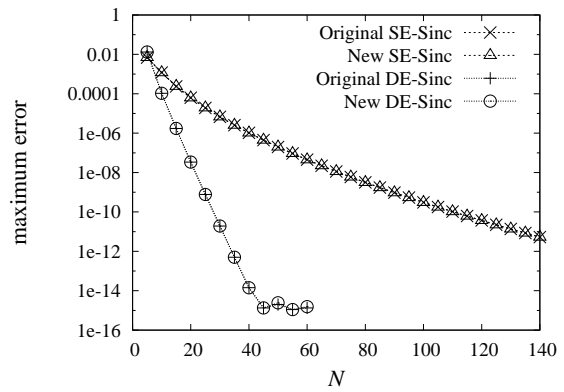


Figure 2: Convergence with respect to N for Example 6.2.

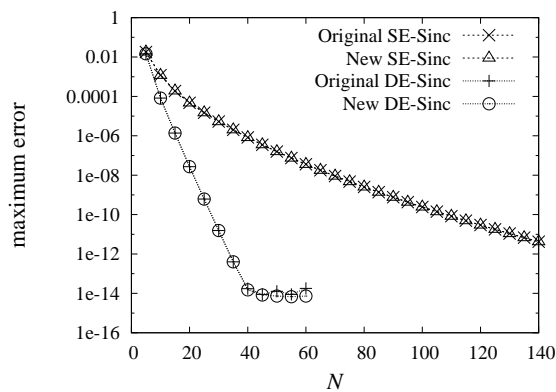


Figure 3: Convergence with respect to N for Example 6.3.

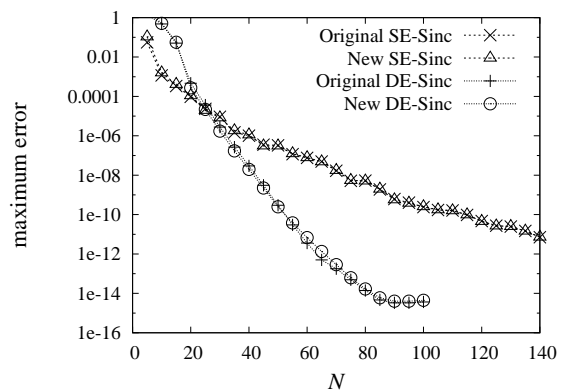


Figure 4: Convergence with respect to N for Example 6.4.

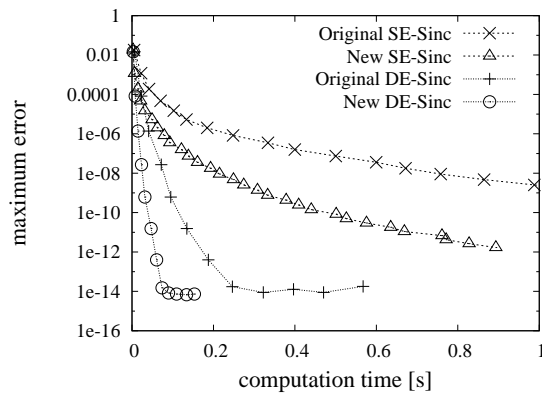


Figure 5: Convergence with respect to computation time for Example 6.3.

7 Proof of convergence theorems

This section provides proofs for Theorems 5.1 and 5.2. In this section, we write $X = C([a, b])$ for short.

7.1 Proof of Theorem 5.1

Recall that by solving (5.4), v_N^{SE} is determined by (5.1), and \tilde{u}_N^{SE} is determined by (5.5). The key point of the proof of Theorem 5.1 is the relation between those two approximate solutions: $\tilde{u}_N^{\text{SE}} = \mathcal{P}_N^{\text{SE}} v_N^{\text{SE}}$. Using the relation, we have

$$\|u - \tilde{u}_N^{\text{SE}}\|_X = \|u - \mathcal{P}_N^{\text{SE}} v_N^{\text{SE}}\|_X \leq \|u - \mathcal{P}_N^{\text{SE}} u\|_X + \|\mathcal{P}_N^{\text{SE}} u - \mathcal{P}_N^{\text{SE}} v_N^{\text{SE}}\|_X \leq \|u - \mathcal{P}_N^{\text{SE}} u\|_X + \|\mathcal{P}_N^{\text{SE}}\|_{\mathcal{L}(X,X)} \|u - v_N^{\text{SE}}\|_X. \quad (7.1)$$

Under the assumptions of Theorem 5.1, $u \in \mathbf{M}_\alpha(\psi^{\text{SE}}(\mathcal{D}_d))$ holds by Theorem 3.1. Thus, using Theorem 2.5, we can evaluate the first term in (7.1) as

$$\|u - \mathcal{P}_N^{\text{SE}} u\|_X \leq C_1 \sqrt{N} \exp\left(-\sqrt{\pi d \alpha N}\right)$$

for a constant C_1 independent of N . As for the second term in (7.1), the following result was obtained.

Theorem 7.1 (Okayama et al. [7, Theorem 6.3]). *Suppose that the assumptions in Theorem 3.1 are fulfilled with $\mathcal{D} = \psi^{\text{SE}}(\mathcal{D}_d)$ for $d \in (0, \pi)$. Then, there exists a positive integer N_0 such that for all $N \geq N_0$, the equation (5.1) has a unique solution $v_N^{\text{SE}} \in X$. Furthermore, there exists a constant C_2 independent of N such that for all $N \geq N_0$,*

$$\|u - v_N^{\text{SE}}\|_X \leq C_2 \exp\left(-\sqrt{\pi d \alpha N}\right).$$

In addition to providing the estimate of $\|u - v_N^{\text{SE}}\|_X$, this theorem guarantees the existence of the unique solution v_N^{SE} ; this means that the system (5.4) is uniquely solvable. Furthermore, we can estimate the remaining term $\|\mathcal{P}_N^{\text{SE}}\|_{\mathcal{L}(X,X)}$ as follows.

Lemma 7.2. *There exists a constant C_3 independent of N such that $\|\mathcal{P}_N^{\text{SE}}\|_{\mathcal{L}(X,X)} \leq C_3 \log(N+1)$.*

The lemma is deduced immediately by using the following result.

Lemma 7.3 (Stenger [13, p. 142]). *Let $h > 0$. Then, it holds that*

$$\sup_{x \in \mathbb{R}} \sum_{j=-N}^N |S(j, h)(x)| \leq \frac{2}{\pi} (3 + \log N).$$

Thus, we can evaluate the second term in (7.1) as

$$\|\mathcal{P}_N^{\text{SE}}\|_{\mathcal{L}(X,X)} \|u - v_N^{\text{SE}}\|_X \leq C_3 \log(N+1) \cdot C_2 \exp\left(-\sqrt{\pi d \alpha N}\right).$$

Summing up the above results, we establish Theorem 5.1.

7.2 Proof of Theorem 5.2

Recall that by solving (5.8), v_N^{DE} is determined by (5.6), and \tilde{u}_N^{DE} is determined by (5.9). In the proof of Theorem 5.2 as well, the key point is the relation between those two approximate solutions: $\tilde{u}_N^{\text{DE}} = \mathcal{P}_N^{\text{DE}} v_N^{\text{DE}}$. Using the relation, we have

$$\|u - \tilde{u}_N^{\text{DE}}\|_X = \|u - \mathcal{P}_N^{\text{DE}} v_N^{\text{DE}}\|_X \leq \|u - \mathcal{P}_N^{\text{DE}} u\|_X + \|\mathcal{P}_N^{\text{DE}} u - \mathcal{P}_N^{\text{DE}} v_N^{\text{DE}}\|_X \leq \|u - \mathcal{P}_N^{\text{DE}} u\|_X + \|\mathcal{P}_N^{\text{DE}}\|_{\mathcal{L}(X,X)} \|u - v_N^{\text{DE}}\|_X. \quad (7.2)$$

Under the assumptions of Theorem 5.2, $u \in \mathbf{M}_\alpha(\psi^{\text{DE}}(\mathcal{D}_d))$ holds by Theorem 3.1. Thus, using Theorem 2.6, we can evaluate the first term in (7.2) as

$$\|u - \mathcal{P}_N^{\text{DE}} u\|_X \leq C_1 \exp\left\{\frac{-\pi d N}{\log(2dN/\alpha)}\right\}$$

for a constant C_1 independent of N . As for the second term in (7.2), the following result was obtained.

Theorem 7.4 (Okayama et al. [7, Theorem 8.1]). *Suppose that the assumptions in Theorem 3.1 are fulfilled with $\mathcal{D} = \psi^{\text{DE}}(\mathcal{D}_d)$ for $d \in (0, \pi/2)$. Then, there exists a positive integer N_0 such that for all $N \geq N_0$, the equation (5.6) has a unique solution $v_N^{\text{DE}} \in X$. Furthermore, there exists a constant C_2 independent of N such that for all $N \geq N_0$,*

$$\|u - v_N^{\text{DE}}\|_X \leq C_2 \exp\left\{\frac{-2\pi d N}{\log(2dN/\alpha)}\right\}.$$

In addition to providing the estimate of $\|u - v_N^{\text{DE}}\|_X$, this theorem guarantees the existence of the unique solution v_N^{DE} ; this means that the system (5.8) is uniquely solvable. Furthermore, in the same way as Lemma 7.2, we can estimate the remaining term $\|\mathcal{P}_N^{\text{DE}}\|_{\mathcal{L}(X,X)}$ as follows.

Lemma 7.5. *There exists a constant C_3 independent of N such that $\|\mathcal{P}_N^{\text{DE}}\|_{\mathcal{L}(X,X)} \leq C_3 \log(N+1)$.*

Thus, we can evaluate the second term in (7.2) as

$$\|\mathcal{P}_N^{\text{DE}}\|_{\mathcal{L}(X,X)} \|u - v_N^{\text{DE}}\|_X \leq C_3 \log(N+1) \cdot C_2 \exp\left\{\frac{-2\pi d N}{\log(2dN/\alpha)}\right\}.$$

Summing up the above results, we establish Theorem 5.2.

8 Concluding remarks

Okayama et al. [7] presented the SE/DE-Sinc-collocation methods for the equation (1.1). Those methods can attain exponential convergence even if the functions k and g have derivative singularity at the endpoints. However, those methods are challenging or painful to implement owing to the inconsistent collocation points (4.3) or (4.8). To address the situation, this paper proposed new SE/DE-Sinc-collocation methods that employ consistent collocation points (5.2) or (5.7). The results of a theoretical analysis proved that the new SE/DE-Sinc-collocation methods exhibit the same convergence property as the original SE/DE-Sinc-collocation methods. As shown in Section 6, compared with the original methods, the number of lines of the programs in the new methods was reduced to 70–80 %. Furthermore, the advantage of the new methods in terms of the computational cost may be observed in the equations with computationally complex functions k and g . It should also be emphasized that the condition numbers of the resulting matrices in the new methods ($I_m - \tilde{K}_m^{\text{SE}}$ and $I_m - \tilde{K}_m^{\text{DE}}$) are good [8], although such a result is not obtained yet for the original methods.

The followings issues may be considered in future studies. First, a Sinc-collocation method for Volterra integral equations was proposed by Rashidinia and Zarebnia [12], and results similar to those of the present work can be established for that method. Second, the error analysis for the Sinc-Nyström methods can be improved compared with Theorems 7.1 and 7.4. As described in Remark 1, the mesh size h for the SE/DE-Sinc quadrature is not selected optimally because it can share the same h as in the SE/DE-Sinc approximation, which is also used in the Sinc-collocation methods. However, in the case of the Sinc-Nyström methods, only the SE/DE-Sinc quadrature is used, and the mesh size h can be optimally chosen. The error of this case should be rigorously analyzed for both cases of the SE transformation [10] and DE transformation [3]. Third, explicit error bounds are desired for verified numerical computation. If we can compute the constant C in Theorem 5.1 or 5.2, we can guarantee the accuracy of the method by computing the right-hand side of the inequality. This can be performed by examining the constants appearing in the convergence theorems, and using the results for basic approximation formulas [6]. We plan to report further results on these issues in future work.

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