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The representation of the limit of power series of positive linear operators by using the semigroup of operators generated by their iterates

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Dedicated to Professor Ioan Raşa on the occasion of his 70th anniversary

Abstract

The limit of power series of a class of positive linear operators is studied using the C_0-semigroup generated by the iterates of these operators.

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1 Introduction

For an operator L, denote by L^k its iterates: $L^k = L \circ \cdots \circ L$, k times, if $k \geq 1$ with $L^0 = I$, being the identity operator.

If $(L_n)_n$, $L_n : C[0,1] \to C[0,1]$ is a sequence of positive linear operators, the geometric series of L_n is of the form

$$\beta_n \sum_{k=0}^{\infty} (L_n)^k,\ n \in \mathbb{N} \tag{1}$$

where $\beta_n \in \mathbb{R}$ is a normalization factor. It is obvious that the geometric series is not defined for any function in $C[0,1]$. For instance, with the hypothesis that L_n preserve constant functions, then the operators in (1) are not defined for such functions. In order to define this geometric series of operators it is necessary to restrict the domain of definition of operators. A space that can be taken in consideration is

$$\psi C[0,1] = \{f \mid \exists g \in C[0,1],\ f = \psi \cdot g\}. \tag{2}$$

where $\psi(x) = x(1-x)$ and the norm on $\psi C[0,1]$ is given by

$$\|\psi g\|_{\psi} = \|g\|,\ \psi g \in \psi C[0,1], \tag{3}$$

where $\|\cdot\|$ denotes the uniform norm.

A first study of the convergence of geometric series attached to a sequence of operators $(L_n)_n$ was made in paper [13], namely the case when L_n are the Bernstein operators, B_n. Here it is shown that one can define operators $A_n : \psi C[0,1] \to \psi C[0,1]$,

$$A_n = \frac{1}{n} \sum_{k=0}^{\infty} (B_n)^k,\ n \in \mathbb{N}$$

and this sequence has a limit when $n \to \infty$ in the space $(\psi C[0,1], \|\cdot\|_{\psi})$, which can be explicitly described.

In this direction several papers extended this study for diverse classes of positive linear operators and for other spaces of functions, see [1], [2], [3], [10], [13], [15], [16].

Recently, in the paper by Acar, Aral and Raşa, [4] it is given a new way to describe the uniform limit of geometric series of form (1) using the semigroup of operators generated by the iterates of L_n.

Our aim is to study the convergence of more general power series of the form:

$$\sum_{k=0}^{\infty} \beta_{n,k} (L_n)^k \tag{4}$$

using the C_0-semigroup generated by the iterates of operators L_n. The framework of our approach differs from the study made in [4] in the sense that we consider another type of operators, another space of functions and a stronger type of convergence.

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2 Preliminaries

Denote $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Denote by Π the set of polynomials and for $r \in \mathbb{N}_0$, denote by Π_r, the space of polynomials of degree at most r. The monomial functions are given by $e_j(x) = x^j$, $(x \in [0,1],\ j \in \mathbb{N}_0)$. Also, if $y \in \mathbb{R}$ and $r \in \mathbb{N}_0$, denote by $(y)_r$, the falling factorial: $(y)_r = y(y-1)\ldots(y-r+1)$, with $(y)_0 = 1$.

Besides the space $\psi C[0,1]$, given in (2), we consider the space, (see [3]):

$$C_\psi[0,1] = \{f \mid \exists g \in B[0,1] \cap C(0,1),\ f = \psi g\}. \tag{5}$$

This is an extension of space $\psi C[0,1]$ and it can be endowed with the same norm:

$$\|\psi g\|_\psi = \|g\|,\ \psi g \in C_\psi[0,1]. \tag{6}$$

This norm can be also defined as

$$\|f\|_\psi = \sup_{x\in(0,1)} \frac{|f(x)|}{x(1-x)},\ f \in C_\psi[0,1]. \tag{7}$$

One can write also,

$$C_\psi[0,1] = \{f \in C[0,1],\ \|f\|_\psi < \infty\}.$$

The space $C_\psi[0,1]$ endowed with the norm $\|\cdot\|_\psi$ is a Banach space, but it is not a Banach space with regard to the sup-norm $\|\cdot\|$, since

$$\overline{\psi C[0,1]} = \overline{C_\psi[0,1]} = C_0[0,1],$$

where $C_0[0,1] = \{f \in C[0,1] | f(0) = 0, f(1) = 0\}$.

Note that if $f, f_n \in C_\psi[0,1]$, $n \in \mathbb{N}$ and $\|f - f_n\|_\psi \to 0$, $(n \to \infty)$ then $\|f - f_n\| \to 0$, $(n \to \infty)$. For this reason, we can name $\|\cdot\|_\psi$ the strong norm on the space $C_\psi[0,1]$.

If $L : C_\psi[0,1] \to C_\psi[0,1]$ is a linear bounded operator we will use the notation

$$\|L\|_\psi = \sup_{\|f\|_\psi \le 1} \|Lf\|_\psi. \tag{8}$$

In the sequel we consider a sequence $(L_n)_n$ of positive linear operators $L_n : C[0,1] \to C[0,1]$, $L_n \ne I$, satisfying the following conditions.

A1) There exist $\alpha \in (0,1)$ and $\alpha_n \in (0,1)$, $n \in \mathbb{N}$ such that $L_n(\psi) = (1-\alpha_n)\psi$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} n\alpha_n = \alpha$.

A2) The operator L_n admits the eigenvalues $a_{n,j}$ associated to eigenpolynomials $p_{n,j}$, $0 \le j \le n$, with $\deg p_{n,j} = j$, where, for a polynomial p we denote by $\deg p$, the degree of p.

A3) There exist polynomials p_j, $j \ge 0$, such that $\lim_{n\to\infty} p_{n,j} = p_j$, $j = 0, 1, \ldots$.

A4) For any $j \ge 0$ there exists $l_j \in (0,1]$, such that

$$\lim_{n\to\infty} (a_{n,j})^n = l_j$$

and moreover if $l_j = 1$, then $a_{n,j} = 1$, for all $n \in \mathbb{N}$.

A5) We have $L_n(\psi\Pi) \subset \psi\Pi$.

A6) There exists a C_0 - semigroup of operators $(T(t))_{t>0}$, such that

$$T(t)f = \lim_{n\to\infty} (L_n)^{k_n} f,\ \text{uniformly for } f \in C[0,1],\ t \ge 0, \tag{9}$$

if $k_n \in \mathbb{N}$, $\lim_{n\to\infty} \frac{k_n}{n} = t$.

From conditions A1)-A6) one can deduce the following consequences.

Remark 1. Because L_n is a positive linear operator and $L_n \ne I$, from condition A4) there are at most two values of $j \ge 0$, for which $l_j = 1$.

Remark 2. Condition A4) implies that

$$\lim_{n\to\infty} a_{n,j} = 1,\ \text{and}\ \lim_{n\to\infty} n(1 - a_{n,j}) = -\ln l_j,\ j = 0, 1, \ldots. \tag{10}$$

Remark 3. Conditions A3), A4) and A6) imply

$$T(t)p_j = l_j^t p_j,\ j \in \mathbb{N}_0,\ t \ge 0. \tag{11}$$

Note that condition A6) is assured in certain hypothesis by Trotter's theorem, ([17]).

Remark 4. For $r \ge 0$, because the polynomials $p_{n,j}$, $0 \le j \le r$ have the property $\deg p_{n,j} = j$, they form a basis of Π_r and consequently $L_n(\Pi_r) \subset \Pi_r$. Then, by induction, $L_n^k(\Pi_r) \subset \Pi_r$, $k \in \mathbb{N}$, for any $n, k \in \mathbb{N}$. From condition A6) it results that $T(t)(\Pi_r) \subset \Pi_r$, $r \in \mathbb{N}_0$.

Remark 5. We mention that the first part of condition A1) is a consequence of the following conditions: $L_n(e_j) = e_j$, $j = 0, 1$ and $L_n(\Pi_2) \subset \Pi_2$. Indeed, it is proved in [3] that if $L : C[0,1] \to C[0,1]$ is a positive linear operator such that $L_n(e_j) = e_j$, $j = 0, 1$ and $L_n(\Pi_2) \subset \Pi_2$, then there exists $\beta \in [0,1)$ such that $L\psi = \beta\psi$.

Also, condition A5) is a consequence of the following conditions: $L_n(C[0,1]) \subset \Pi$ and $L_n(e_j) = e_j$, $j = 0, 1$. Indeed, in this case we have $L_n f(0) = f(0)$ and $L_n f(1) = f(1)$, for any $f \in C[0,1]$. Consequently, for $f \in \psi C[0,1]$ it follows that $L_n f(0) = L_n f(1) = 0$ and hence $L_n(\psi C[0,1]) \subset \psi\Pi$.

Finally we need the following lemmas.

Lemma 2.1. *For any $t \geq 0$ one has $T(t)(C_\psi[0,1]) \subset C_\psi[0,1]$ and*

$$\|T(t)\|_\psi = e^{-\alpha t}. \tag{12}$$

Proof. Let $f \in C_\psi[0,1]$. Since $|f| \leq \|f\|_\psi \psi$, one obtains $|L_n f| \leq L_n|f| \leq \|f\|_\psi L_n\psi = (1-\alpha_n)\|f\|_\psi\psi$. Hence, by induction we have for $k \in \mathbb{N}$ that $|L_n^k f| \leq (1-\alpha_n)^k \|f\|_\psi \psi$. Take $k = k_n$, such that $k_n/n \to t$. We get

$$\frac{|(L_n^{k_n} f)(x)|}{\psi(x)} \leq (1-\alpha_n)^{k_n} \|f\|_\psi, \; x \in (0,1).$$

By passing to the limit as $n \to \infty$ it follows that

$$\frac{|(T(t)f)(x)|}{\psi(x)} \leq \lim_{n\to\infty} (1-\alpha_n)^{k_n} \|f\|_\psi = e^{-\alpha t}\|f\|_\psi.$$

Then $T(t)f \in C_\psi[0,1]$ and $\|T(t)\|_\psi \leq e^{-\alpha t}$. Finally, taking $f = \psi$ one gets $T(t)\psi = \lim_{n\to\infty}(1-\alpha_n)^{k_n}\psi = e^{-\alpha t}\psi$. So, $\|T(t)\|_\psi = e^{-\alpha t}$. □

Lemma 2.2. *Let $r \in \mathbb{N}$. If a sequence of polynomials $(\sigma_n)_n$, $\sigma_n \in \psi\Pi_r$ is uniformly convergent to a polynomial $\sigma^\star \in \psi\Pi_r$, then sequence $(\sigma_n)_n$ converges to $\sigma^\star$ in the norm $\|\cdot\|_\psi$ as well.*

Proof. Let $q_n, q^\star \in \Pi_r$, $n \in \mathbb{N}$ be such that $\sigma_n = \psi q_n$ and $\sigma^\star = \psi q^\star$. Because σ_n converges uniformly to $\sigma^\star$, the coefficients of σ_n converge to the corresponding coefficients of $\sigma^\star$. But this implies that the coefficients of q_n converge to the corresponding coefficients of $q^\star$. This means that $(q_n)_n$ converges uniformly to $q^\star$, which is equivalent to the fact that $(\sigma_n)_n$ converges in norm $\|\cdot\|_\psi$ to $\sigma^\star$. □

3 Main results

A main tool for our purpose is the following lemma.

Lemma 3.1. *For $p \in \Pi$, $s \in \mathbb{N}_0$, $t \geq 0$ and a sequence of positive integers $(k_n)_n$ such that $k_n/n \to t$, $(n \to \infty)$ there exists the limit:*

$$\lim_{n\to\infty} \frac{1}{k_n^{s+1}} \sum_{i=0}^{k_n} (i)_s (L_n)^i p = \frac{1}{t^{s+1}} \int_0^t u^s T(u) p du. \tag{13}$$

uniformly, where $(i)_s = i(i-1)\ldots(i-s+1)$.

Proof. First, let us remark that if we take $p \in \Pi_r \setminus \Pi_{r-1}$, then we can write $(n \geq r)$:

$$p = \sum_{j=0}^{r} \gamma_{n,j} p_{n,j}, \text{ with } \gamma_{n,j} \in \mathbb{R},\ \gamma_{n,r} \neq 0.$$

This means that

$$\begin{aligned} \sum_{i=0}^{k_n} (i)_s (L_n)^i p &= \sum_{i=0}^{k_n} (i)_s (L_n)^i \sum_{j=0}^{r} \gamma_{n,j} p_{n,j} \\ &= \sum_{j=0}^{r} \gamma_{n,j} \left(\sum_{i=0}^{k_n} (i)_s (a_{n,j})^i \right) p_{n,j}. \end{aligned} \tag{14}$$

We show that there exist the finite limits

$$\eta_j^s := \lim_{n\to\infty} \frac{1}{k_n^{s+1}} \sum_{i=0}^{k_n} (i)_s (a_{n,j})^i, \text{ for } j, s \in \mathbb{N}_0. \tag{15}$$

First we will need a formula for the sum $\sum_{i=0}^{k} (i)_s x^i$, $x \neq 1$. We have that

$$\begin{aligned}\sum_{i=0}^{k}(i)_s x^i &= \sum_{i=0}^{k} x^s \frac{d^s}{dx^s}(x^i)\\ &= x^s \frac{d^s}{dx^s}\left(\frac{1-x^{k+1}}{1-x}\right)\\ &= x^s\left(\frac{s!(1-x^{k+1})}{(1-x)^{s+1}}-\sum_{\nu=1}^{s}\binom{s}{\nu}(k+1)_\nu x^{k+1-\nu}\frac{(s-\nu)!}{(1-x)^{s-\nu+1}}\right),\end{aligned}$$

where the sum is null in the case $s=0$.

First we consider that $j\geq 0$ is such that $l_j\in(0,1)$. Take n sufficiently large such that $a_{n,j}\in(0,1)$. Now, replacing x by $a_{n,j}$ and k by k_n in the formula from above and then using condition A4) and the limit (10) we get:

$$\begin{aligned}\eta_j^s &= \lim_{n\to\infty} a_{n,j}^s \frac{n^{s+1}}{k_n^{s+1}}\left(\frac{s!(1-(a_{n,j}^n)^{\frac{k_n+1}{n}})}{[n(1-a_{n,j})]^{s+1}}\right.\\ &\qquad\left.-\sum_{\nu=1}^{s}\binom{s}{\nu}(k_n+1)_\nu a_{n,j}^{k_n+1-\nu}\frac{(s-\nu)!}{[n(1-a_{n,j})^{s-\nu+1}]n^\nu}\right)\\ &= \frac{1}{t^{s+1}}\left[\frac{s!(1-l_j^t)}{(-\ln l_j)^{s+1}}-\sum_{\nu=1}^{s}\frac{s!}{\nu!}\frac{l_j^t}{(-\ln l_j)^{s-\nu+1}}t^\nu\right]\\ &= \frac{1}{t^{s+1}}\left[\frac{s!}{(-\ln l_j)^{s+1}}-\sum_{\nu=0}^{s}\frac{s!}{\nu!}\frac{l_j^t}{(-\ln l_j)^{s-\nu+1}}t^\nu\right].\end{aligned} \tag{16}$$

Using Taylor's polynomial with integral remainder for function $f(t)=l_j^t/(\ln l_j)^\beta$, with $\beta\in\mathbb{N}$ one has:

$$\frac{l_j^t}{(\ln l_j)^\beta}=\sum_{\mu=0}^{\beta-1}\frac{t^\mu}{\mu!}\frac{1}{(\ln l_j)^{\beta-\mu}}+\int_0^t\frac{(t-u)^{\beta-1}}{(\beta-1)!}l_j^u du. \tag{17}$$

Using formula (17), we can see that

$$\begin{aligned}&\frac{s!}{(-\ln l_j)^{s+1}}-\sum_{\nu=0}^{s}\frac{s!}{\nu!}\frac{l_j^t}{(-\ln l_j)^{s-\nu+1}}t^\nu\\ &=(-1)^{s+1}\frac{s!}{(\ln l_j)^{s+1}}-\sum_{\nu=0}^{s}(-1)^{s-\nu+1}\frac{s!}{\nu!}t^\nu\sum_{\mu=0}^{s-\nu}\frac{t^\mu}{\mu!}\frac{1}{(\ln l_j)^{s-\nu-\mu+1}}\\ &\qquad-\sum_{\nu=0}^{s}(-1)^{s-\nu+1}\frac{s!}{\nu!}t^\nu\int_0^t\frac{(t-u)^{s-\nu}}{(s-\nu)!}l_j^u du.\end{aligned} \tag{18}$$

But, using the substitution $\zeta=\mu+\nu$ one obtains

$$\begin{aligned}&(-1)^{s+1}\frac{s!}{(\ln l_j)^{s+1}}-\sum_{\nu=0}^{s}(-1)^{s-\nu+1}\frac{s!}{\nu!}t^\nu\sum_{\mu=0}^{s-\nu}\frac{t^\mu}{\mu!}\frac{1}{(\ln l_j)^{s-\nu-\mu+1}}\\ &= (-1)^{s+1}\frac{s!}{(\ln l_j)^{s+1}}+\sum_{\nu=0}^{s}\sum_{\mu=0}^{s-\nu}(-1)^{s-\nu}\frac{t^{\nu+\mu}}{\nu!\mu!}\frac{s!}{(\ln l_j)^{s-\nu-\mu+1}}\\ &= (-1)^{s+1}\frac{s!}{(\ln l_j)^{s+1}}+\sum_{\zeta=0}^{s}(-1)^{s-\zeta}\frac{t^\zeta}{\zeta!}\frac{s!}{(\ln l_j)^{s-\zeta+1}}\sum_{\mu=0}^{\zeta}(-1)^\mu\binom{\zeta}{\mu}\\ &= (-1)^{s+1}\frac{s!}{(\ln l_j)^{s+1}}+(-1)^s\frac{s!}{(\ln l_j)^{s+1}}\\ &= 0.\end{aligned} \tag{19}$$

So, from (16), (18) and (19) we obtain

$$\begin{aligned}\eta_j^s &= \frac{1}{t^{s+1}}\sum_{\nu=0}^{s}(-1)^{s-\nu}\frac{s!}{\nu!}t^\nu\int_0^t\frac{(t-u)^{s-\nu}}{(s-\nu)!}l_j^u du\\ &= \frac{1}{t}\int_0^t\sum_{\nu=0}^{s}(-1)^{s-\nu}\binom{s}{s-\nu}\left(\frac{t-u}{t}\right)^{s-\nu}l_j^u du\\ &= \frac{1}{t}\int_0^t\left(\frac{u-t}{t}+1\right)^s l_j^u du\\ &= \frac{1}{t^{s+1}}\int_0^t u^s l_j^u du.\end{aligned}$$

In addition, in the case when $j \geq 0$ is such that $l_j = 1$, it follows from condition A4) that $a_{n,j} = 1$ for all $n \in \mathbb{N}$. By a simple computation we get $\eta_j^s = 1/(s+1)$. Then we conclude that for all $j, s \in \mathbb{N}_0$ one has:

$$\eta_j^s = \frac{1}{t^{s+1}} \int_0^t u^s l_j^u du. \tag{20}$$

From (15) and (20) it follows

$$\lim_{n\to\infty} \frac{1}{k_n^{s+1}} \sum_{i=0}^{k_n} (i)_s (a_{n,j})^i = \frac{1}{t^{s+1}} \int_0^t u^s l_j^u du, \text{ for } j, s \in \mathbb{N}_0 \tag{21}$$

On the other hand we can also represent p as $p = \sum_{j=0}^r \gamma_j p_j$, where polynomials p_j are given in A3). There exist the following limits:

$$\lim_{n\to\infty} \gamma_{n,j} = \gamma_j, \ 0 \leq j \leq r. \tag{22}$$

Indeed, since $p_{n,j}$ converges uniformly to p_j it follows that all the coefficients of $p_{n,j}$ converge to the corresponding coefficients of p_j. If q is a polynomial, we denote by $\text{coeff}_s(q)$ the coefficient of x^s in q. One has $\text{coeff}_r(\gamma_{n,r} p_{n,r}) = \text{coeff}_r(p) = \text{coeff}_r(\gamma_r p_r)$. Because $\text{coeff}_r(p_{n,r}) \neq 0$, $\text{coeff}_r(p_r) \neq 0$, and $\text{coeff}_r(p_{n,r}) \to \text{coeff}_r(p_r)$ it follows that $\gamma_{n,r} \to \gamma_r$. Then one can proceed by induction. Suppose that $\gamma_{n,j} \to \gamma_j$, for $s \leq j \leq r$, $s \geq 1$. We have $\text{coeff}_{s-1}(\gamma_{n,s-1} p_{n,s-1})= \text{coeff}_{s-1}\left(\sum_{j=0}^{s-1} \gamma_{n,j} p_{n,j}\right) = \text{coeff}_{s-1}\left(p - \sum_{j=s}^r \gamma_{n,j} p_{n,j}\right)$ $\to \text{coeff}_{s-1}\left(p - \sum_{j=s}^r \gamma_j p_j\right) = \text{coeff}_{s-1}\left(\sum_{j=0}^{s-1} \gamma_j p_j\right) = \text{coeff}_{s-1}(\gamma_{s-1} p_{s-1})$. From this it follows $\gamma_{n,s-1} \to \gamma_{s-1}$. Relation (22) is proved.

Therefore, taking into account condition A3), (14), (21) and (22) it follows that

$$\lim_{n\to\infty} \frac{1}{k_n^{s+1}} \sum_{i=0}^{k_n} (i)_s (L_n)^i p = \frac{1}{t^{s+1}} \sum_{j=0}^{r} \gamma_j \int_0^t u^s l_j^u du \cdot p_j. \tag{23}$$

On the other hand, using (11) we will obtain

$$\int_0^t u^s T(u) p \, du = \int_0^t u^s T(u) \left(\sum_{j=0}^r \gamma_j p_j \right) du = \sum_{j=0}^r \gamma_j \int_0^t u^s l_j^u du \cdot p_j. \tag{24}$$

So, from (23) and (24) we have that

$$\lim_{n\to\infty} \frac{1}{k_n^{s+1}} \sum_{i=0}^{k_n} (i)_s (L_n)^i p = \frac{1}{t^{s+1}} \int_0^t (T(u)p) u^s du,$$

which ends our proof. □

Corollary 3.2. *For any $p \in \psi\Pi$, $s \in \mathbb{N}_0$, $t > 0$ and a sequence of positive integers $(k_n)_n$ such that $k_n/n \to t$, $(n \to \infty)$ there holds*

$$\lim_{n\to\infty} \left\| \frac{1}{k_n^{s+1}} \sum_{i=0}^{k_n} (i)_s (L_n)^i p - \frac{1}{t^{s+1}} \int_0^t u^s T(u) p du \right\|_\psi = 0. \tag{25}$$

Proof. Let $p \in \psi\Pi_{r-2}$, $r \geq 2$ and also $t > 0$, $s \in \mathbb{N}_0$ and $(k_n)_n$, like in Lemma 3.1. From condition A5) and Remark 4 there exist $q_n \in \Pi_{r-2}$, $n \in \mathbb{N}_0$ and $q^\star \in \Pi_{r-2}$ such that $k_n^{-s-1} \sum_{i=0}^{k_n} (i)_s (L_n)^i p = \psi q_n$ and $t^{-s-1} \int_0^t u^s T(u) p du = \psi q^\star$. From Lemma 3.1, the sequence $(\psi q_n)_n$ converges uniformly to $\psi q^\star$ and from Lemma 2.2 the sequence $(\psi q_n)_n$ converges in norm $\|\cdot\|_\psi$ to $\psi q^\star$. This means that (25) is satisfied. □

Now we need the following theorem, which with modified notations, follows from a result proved in the book of Nachbin [12], see Lemma 2, pg. 95.

Theorem A *Let $b > 0$. For any function $f \in C[0,\infty)$, such that $f(x)e^{-bx} \to 0$, $(x \to \infty)$, and any $\varepsilon > 0$ there exist a polynomial p such that*

$$\sup_{x\in[0,\infty)} e^{-bx} |f(x) - p(x)| < \varepsilon.$$

In the terminology from [12], the function e^{-bx}, $x \geq 0$ is a fundamental weight.

Define the space:

$$\tilde{C}_\alpha[0,\infty) = \{g \in C[0,\infty) | \ \exists b \in (0,\alpha), \ \lim_{x\to\infty} g(x)e^{-bx} = 0\}. \tag{26}$$

Our main result is the following:

Theorem 3.3. *If $g \in \tilde{C}_\infty[0,\infty)$ and $f \in \psi C[0,1]$ then*

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{i=0}^{\infty} g\left(\frac{i}{n}\right) (L_n)^i f - \int_0^\infty g(t) T(t) f \, dt \right\|_\psi = 0. \tag{27}$$

Proof. From Lemma 2.1 we obtain

$$\int_0^\infty \|g(t)T(t)f\|_\psi dt \le \int_0^\infty |g(t)|\cdot\|T(t)\|_\psi \|f\|_\psi dt = \|f\|_\psi \int_0^\infty |g(t)|e^{-\alpha t}dt < \infty.$$

It follows that the integral in (27) exists and is absolutely convergent.

First, we will prove that, for a fixed $m \in \mathbb{N}$ and $p \in \psi\Pi$ we have:

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{i=0}^{\infty}\left(\frac{i}{n}\right)^m (L_n)^i p - \int_0^\infty u^m T(u)p du\right\|_\psi = 0. \tag{28}$$

Let $\varepsilon > 0$. Because the integral from (28) is convergent we can fix a number $t > 0$ such that

$$\left\|\int_t^\infty u^m T(u)p du\right\|_\psi < \varepsilon. \tag{29}$$

Also we have

$$\frac{1}{n}\sum_{i=0}^{\infty}\left(\frac{i}{n}\right)^m \|(L_n)^i p\|_\psi \le \frac{1}{n}\sum_{i=0}^{\infty}\left(\frac{i}{n}\right)^m (1-\alpha_n)^i \|p\|_\psi < \infty,\ n \in \mathbb{N}.$$

Then the series in (28) is absolutely convergent.

Take a sequence of positive integers $(k_n)_n$ such that $k_n/n \to t$, $(n \to \infty)$. Consequently there exists $n_\varepsilon^1 \in \mathbb{N}$, such that

$$\left\|\frac{1}{n}\sum_{i=k_n+1}^{\infty}\left(\frac{i}{n}\right)^m (L_n)^i p\right\|_\psi < \varepsilon,\ \text{for } n \ge n_\varepsilon^1. \tag{30}$$

For $n \ge n_\varepsilon^1$ one has

$$\begin{aligned}
&\left\|\frac{1}{n}\sum_{i=0}^{k_n}\left(\frac{i}{n}\right)^m (L_n)^i p - \int_0^t u^m T(u)p du\right\|_\psi \\
\le\ & \frac{1}{n^{m+1}}\left\|\sum_{i=0}^{k_n} i^m (L_n)^i p - \sum_{i=0}^{k_n}(i)_m (L_n)^i p\right\|_\psi \\
&+\left|\left(\frac{k_n}{n}\right)^{m+1} - t^{m+1}\right|\cdot\left\|\frac{1}{k_n^{m+1}}\sum_{i=0}^{k_n}(i)_m (L_n)^i p\right\|_\psi \\
&+t^{m+1}\left\|\frac{1}{k_n^{m+1}}\sum_{i=0}^{k_n}(i)_m (L_n)^i p - \frac{1}{t^{m+1}}\int_0^t u^m T(u)p du\right\|_\psi \\
=:\ & T_1^n + T_2^n + T_3^n.
\end{aligned}$$

Using Stirling numbers $s_{m,j}$, $0 \le j \le m$, i.e. $i^m = \sum_{j=0}^m s_{m,j}(i)_j$ we can write

$$\begin{aligned}
T_1^n &= \left(\frac{k_n}{n}\right)^{m+1}\left\|\frac{1}{k_n^{m+1}}\sum_{i=0}^{k_n}\sum_{j=0}^{m} s_{m,j}(i)_j (L_n)^i p - \frac{1}{k_n^{m+1}}\sum_{i=0}^{k_n}(i)_m (L_n)^i p\right\|_\psi \\
&= \left(\frac{k_n}{n}\right)^{m+1}\left\|\frac{1}{k_n^{m+1}}\sum_{i=0}^{k_n}\sum_{j=0}^{m-1} s_{m,j}(i)_j (L_n)^i p\right\|_\psi \\
&\le \left(\frac{k_n}{n}\right)^{m+1}\sum_{j=0}^{m-1}\frac{s_{m,j}}{k_n^{m-j}}\left\|\frac{1}{k_n^{j+1}}\sum_{i=0}^{k_n}(i)_j (L_n)^i p\right\|_\psi.
\end{aligned}$$

Then, from Corollary 3.2, $\lim_{n\to\infty} T_1^n = 0$, since $(k_n/n)^{m+1} \to t^{m+1}$ and $k_n^{-j-1}\sum_{i=0}^{k_n}(i)_j (L_n)^i p$ is bounded in $\|\cdot\|_\psi$ norm. Also, using again Corollary 3.2, for $s = m$ and $(k_n/n)^{m+1} \to t^{m+1}$ we get $\lim_{n\to\infty} T_2^n = 0$ and $\lim_{n\to\infty} T_3^n = 0$, Consequently there exists $n_\varepsilon^2 \in \mathbb{N}$, $n_\varepsilon^2 > n_\varepsilon^1$, such that

$$\left\|\frac{1}{n}\sum_{i=0}^{k_n}\left(\frac{i}{n}\right)^m (L_n)^i p - \int_0^t u^m T(u)p du\right\|_\psi < \varepsilon,\ \text{for } n \ge n_\varepsilon^2. \tag{31}$$

Then (28) follows from (29), (30) and (31).

Now, let $f \in \psi C[0,1]$ and $\varepsilon > 0$. Then there is $p \in \psi\Pi$ such that $\|f - p\|_\psi < \varepsilon$. Like in the proof of Lemma 2.1 we deduce that $\|(L_n)^i\|_\psi = (1-\alpha_n)^i$, for $i \in \mathbb{N}_0$. We have that

$$\begin{aligned}
& \left\| \frac{1}{n}\sum_{i=0}^{\infty}\left(\frac{i}{n}\right)^m (L_n)^i f - \frac{1}{n}\sum_{i=0}^{\infty}\left(\frac{i}{n}\right)^m (L_n)^i p \right\|_\psi \\
\le\; & \frac{1}{n}\sum_{i=0}^{\infty}\left(\frac{i}{n}\right)^m \|(L_n)^i\|_\psi \|f-p\|_\psi \\
<\; & \varepsilon \frac{1}{n}\sum_{i=0}^{\infty}\left(\frac{i}{n}\right)^m (1-\alpha_n)^i \\
=\; & \varepsilon \frac{1}{n^{m+1}}\sum_{i=0}^{\infty}\sum_{j=0}^{m} s_{m,j}(i)_j(1-\alpha_n)^i \\
=\; & \varepsilon \sum_{j=0}^{m} s_{m,j}\frac{1}{n^{m+1}}(1-\alpha_n)^j \sum_{i=j}^{\infty}(i)_j(1-\alpha_n)^{i-j} \\
=\; & \varepsilon \sum_{j=0}^{m} s_{m,j}(1-\alpha_n)^j \frac{1}{n^{m+1}}\frac{j!}{(\alpha_n)^{j+1}}.
\end{aligned}$$

Condition A1) implies that $\lim_{n\to\infty} n\alpha_n = \alpha$. Hence there is $M > 0$ and $n_\varepsilon \in \mathbb{N}$ such that

$$\left\| \frac{1}{n}\sum_{i=0}^{\infty}\left(\frac{i}{n}\right)^m (L_n)^i f - \frac{1}{n}\sum_{i=0}^{\infty}\left(\frac{i}{n}\right)^m (L_n)^i p \right\|_\psi < \varepsilon M,\ n \ge n_\varepsilon. \tag{32}$$

Also,

$$\begin{aligned}
\left\| \int_0^\infty u^m T(u) p\, du - \int_0^\infty u^m T(u) f\, du \right\|_\psi &\le \int_0^\infty u^m \|T(u)(p-f)\|_\psi du \\
&\le \varepsilon \int_0^\infty u^m \|T(u)\|_\psi du \le \varepsilon \int_0^\infty u^m e^{-\alpha u} du.
\end{aligned} \tag{33}$$

From (28), (32) and (33), since for a given m, the number $\varepsilon > 0$ can be chosen arbitrarily, one obtains

$$\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{i=0}^{\infty}\left(\frac{i}{n}\right)^m (L_n)^i f - \int_0^\infty u^m T(u) f\, du \right\|_\psi = 0, \tag{34}$$

for any $m \in \mathbb{N}_0$ and $f \in \psi C[0,1]$. Then, from (34), for any $q \in \Pi$ and $f \in \psi C[0,1]$, we have

$$\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{i=0}^{\infty} q\left(\frac{i}{n}\right)(L_n)^i f - \int_0^\infty q(u) T(u) f\, du \right\|_\psi = 0. \tag{35}$$

Now, let $g \in \tilde{C}_\alpha[0,\infty)$. There exists $b \in (0,\alpha)$, such that $\lim_{x\to\infty} g(x)e^{-bx} = 0$. For $\varepsilon > 0$, from Theorem A there exists $q \in \Pi$ such that

$$|q(x) - g(x)|e^{-bx} < \varepsilon,\ x \in [0,\infty).$$

Therefore, for $f \in \psi C[0,1]$, we have

$$\begin{aligned}
& \left\| \int_0^\infty q(u)T(u) f\, du - \int_0^\infty g(u) T(u) f\, du \right\|_\psi \le \int_0^\infty |g(u) - q(u)| \cdot \|T(u) f\|_\psi du \\
& \le \varepsilon \int_0^\infty e^{bu} \|T(u)\|_\psi \cdot \|f\|_\psi du \\
& = \varepsilon \|f\|_\psi \int_0^\infty e^{bu} e^{-\alpha u} du \\
& = \varepsilon \|f\|_\psi \frac{1}{\alpha - b}.
\end{aligned} \tag{36}$$

Similarly,

$$\begin{aligned}
& \left\| \frac{1}{n}\sum_{i=0}^{\infty} g\left(\frac{i}{n}\right)(L_n)^i f - \frac{1}{n}\sum_{i=0}^{\infty} q\left(\frac{i}{n}\right)(L_n)^i f \right\|_\psi \\
\le\; & \frac{1}{n}\sum_{i=0}^{\infty} \left| g\left(\frac{i}{n}\right) - q\left(\frac{i}{n}\right) \right| \|L_n\|_\psi^i \|f\|_\psi \\
<\; & \frac{\varepsilon}{n}\sum_{i=0}^{\infty} e^{b\frac{i}{n}} (1-\alpha_n)^i \|f\|_\psi \\
=\; & \frac{\varepsilon}{n}\|f\|_\psi \frac{1}{1 - e^{b/n}(1-\alpha_n)}.
\end{aligned}$$

Since $\lim_{n\to\infty} \frac{1}{n(1-e^{b/n}(1-\alpha_n))} = \frac{1}{\alpha-b}$, there is $M > 0$ such that

$$\left\| \frac{1}{n}\sum_{i=0}^{\infty} g\left(\frac{i}{n}\right)(L_n)^i f - \frac{1}{n}\sum_{i=0}^{\infty} q\left(\frac{i}{n}\right)(L_n)^i f \right\|_{\psi} < \varepsilon M. \tag{37}$$

Because $\varepsilon > 0$ can be chosen arbitrarily in (36) and (37), by taking into account (35) we get (23).

□

4 Applications

1. Bernstein operators

Let $B_n : C[0,1] \to C[0,1]$ be the Bernstein operators defined as:

$$(B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right),$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ 0 \le k \le n,\ x \in [0,1].$$

Operators B_n satisfy conditions A1)-A6), see [6], [7] and [11]. More exactly, we have $B_n\psi = \frac{n-1}{n}\psi$ and hence $\alpha_n = 1/n$ and $\alpha = 1$. B_n admits the eigenvalues $a_{n,j}$ corresponding to the eigenpolynomials $p_{n,j}$, $0 \le j \le n$, with $\deg p_{n,j} = j$ and

$$l_j := \lim_{n\to\infty} a_{n,j}^n = e^{-j(j-1)/2},\ j \in \mathbb{N}_0.$$

For $j = 0,1$, we have $p_{n,j}(t) = e_j$ and $B_n(e_j) = e_j$. The existence of the polynomials $p_j = \lim_{n\to\infty} p_{n,j}$ is proved in [7]. Finally the existence of the semigroup of operators generated by the iterates of B_n is given, for instance in [5].

2. Operators U_n^ρ

For $\rho > 0$ and $n \in \mathbb{N}$, $n \ge 2$, operators U_n^ρ are defined, (see [8]), [14], as follows:

$$(U_n^\rho f)(x) = \sum_{k=0}^{n} p_{n,k}(x) F_{n,k}^\rho(f),\ f \in C[0,1],\ x \in [0,1],$$

where

$$F_{n,0}(f) = f(0),\ F_{n,n}(f) = f(1);$$
$$F_{n,k}(f) = \int_0^1 f(t)\frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{B(k\rho,(n-k)\rho)}dt,\ 1 \le k \le n-1.$$

The eigenstructure of these operators was investigated in [9].

These operators also satisfy the conditions A1)-A6). More precisely we have the following: $U_n^\rho\psi = \frac{n-1}{n\rho+1}\psi$, thus we can take $\alpha_n = \frac{\rho+1}{n\rho+1}$ and $\alpha = 1$. Then, U_n^ρ admits the eigenpolynomials $p_{n,j}$, $0 \le j \le n$, with $\deg p_{n,j} = j$. Moreover $p_{n,j} = e_j$ and $U_n^\rho(e_j) = e_j$, for $j = 0,1$. The eigenvalues are, see [9]:

$$a_{n,j} = \rho^j \frac{n(n-1)\dots(n-j+1)}{(n\rho)(n\rho+1)\dots(n\rho+j-1)},\ 0 \le j \le n.$$

Therefore we have

$$l_j := \lim_{n\to\infty} a_{n,j}^n = e^{-\frac{j(j-1)}{2}\cdot\frac{\rho+1}{\rho}},\ j \ge 0.$$

The existence of limit polynomials $p_j = \lim_{n\to\infty} p_{n,j}$ is also shown in [9]. For proving the existence of the semigroup of operators generated by the iterates of operators U_n^ρ one can apply Corollary 2.2.11 from [5].

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