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Positive linear operators preserving certain monomials on $[0,\infty)$

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Abstract

J.P. King constructed a sequence of positive linear operators on $C[0,1]$ preserving the constant function 1 and the monomial x^2. After that, several papers were devoted to positive linear operators preserving prescribed functions. In 2009, J.M. Aldaz, O. Kounchev, H. Render modified the classical Bernstein operators to get new operators fixing 1 and x^j for a given $j \in \mathbb{N}$.
Our aim is to construct operators acting on functions defined on $[0,\infty)$ and preserving 1 and x^j. To this end we consider suitable modifications of the Post-Widder and Gamma operators. Using pointwise asymptotic relations for the moments we obtain convergence results for the sequences of operators. Moreover, we establish the associated Voronovskaja formulas.

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1 Introduction

Many approximation operators L_n preserve linear functions, i.e., $L_n e_r = e_r$, for $r = 0,1$, while $L_n e_2 \neq e_2$ and $L_n e_2 \to e_2$ as $n \to \infty$. For instance, Bernstein operators, Szász-Mirakjan operators and Baskakov operators enjoy this property. It is well-known, that the only positive linear operator satisfying $L_n e_r = e_r$, for $r = 0,1,2$, is the identity operator.

In 2003, J. P. King [14] proposed, for approximation operators $L_n : C[0,1] \to C[0,1]$ preserving linear functions, an operator of the type L_n^*, defined by $\left(L_n^* f\right)(x) = (L_n f)(\tau_n(x))$ with certain continuous functions $\tau_n : [0,1] \to [0,1]$, such that $L_n^* e_r = e_r$, for $r = 0,2$. In this case $L_n^* e_1 = \tau_n$. The Korovkin theorem implies that $\lim_{n\to\infty} L_n^* f = f$, for each $f \in C[0,1]$, if and only if $\lim_{n\to\infty} \tau_n = e_1$ ([14, Theorem 2.1]).

In 2009, J.M. Aldaz, O. Kounchev, H. Render [4] used a new idea to get operators fixing e_0 and e_j for a given $j \in \mathbb{N}$. These operators are linear combinations of the fundamental Bernstein polynomials with modified points of evaluation as follows

$$\left(B_{n,j} f\right)(x) = \sum_{k=0}^{n} f\left(t_{n,k,j}\right) p_{n,k}(x),$$

where

$$t_{n,k,j} = \left(\frac{k(k-1)\ldots(k-j+1)}{n(n-1)\ldots(n-j+1)}\right)^{1/j}.$$

The operator $B_{n,j}$ is one of several examples of operators preserving e_0 and e_j. Operators preserving polynomial functions are investigated in [3], [10], [11], [12], [7], [8]. The preservation of e_0 and a given function τ is studied in [7] and [13].

The aim of this paper is to construct operators acting on the interval $[0,\infty)$ and fixing e_0 and e_j.

We consider two different modifications of the classical Post-Widder and Gamma operators.

The Post-Widder operators indexed by integers $n \geq 1$ (see [9, (9.1.9)]), are given by

$$(P_n f)(x) = \frac{1}{(n-1)!}\left(\frac{n}{x}\right)^n \int_0^\infty e^{-ns/x} s^{n-1} f(s)\,ds,$$

and the Gamma operators (see [9, 9.1.11]) by

$$\begin{aligned}(G_n f)(x) &= \frac{(nx)^{n+1}}{n!} \int_0^\infty \frac{1}{s^{n+2}} e^{-nx/s} f(s)\,ds \\ &= \frac{x^{n+1}}{n!} \int_0^\infty u^n e^{-ux} f\left(\frac{n}{u}\right) du,\end{aligned}$$

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for each function f, for which the respective integral on the right side is convergent.

For $\alpha \in [0,\infty)$ we denote by e_α the function $e_\alpha(x) = x^\alpha$, $x \in [0,\infty)$.

It is easy to confirm that

$$P_n e_\alpha = \frac{\Gamma(n+\alpha)}{n^\alpha \Gamma(n)} e_\alpha,\ \alpha \geq 0 \quad \text{and} \quad G_n e_\alpha = \frac{n^\alpha \Gamma(n+1-\alpha)}{\Gamma(n+1)} e_\alpha, \quad 0 \leq \alpha < n+1. \tag{1}$$

For nonnegative integers $\alpha = r$, we have

$$P_n e_r = \lambda_{n,r} e_r,\ \text{with}\ \lambda_{n,r} := \frac{n^{\overline{r}}}{n^r} \tag{2}$$

and

$$G_n e_r = \mu_{n,r} e_r,\ \text{with}\ \mu_{n,r} := \frac{n^r}{n^{\underline{r}}},\ n \geq r, \tag{3}$$

where $n^{\overline{r}}$, $n^{\underline{r}}$ are the rising and falling factorials. In particular, P_n and G_n preserve the linear functions.

2 Auxiliary results

In this paper we need estimates of the ratio of two gamma functions $\Gamma(z+\alpha)/\Gamma(z+\beta)$, for $z,\alpha,\beta > 0$. It is well known that the gamma ratio satisfies, for any real α, the classical relation $\lim_{z\to+\infty} z^{-\alpha}\Gamma(z+\alpha)/\Gamma(z) = 1$. We apply the following more general asymptotic relation, namely,

$$z^{\beta-\alpha}\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = 1 + \frac{(\alpha-\beta)(\alpha+\beta-1)}{2z} + O\left(\frac{1}{z^2}\right) \tag{4}$$

as $z \to +\infty$ (see, e.g., [1, Formula (6.1.47)]. For our purposes, we also need concrete lower and upper bounds for the gamma ratio. We take advantage of the following estimate due to Wendel [15].

Lemma 2.1. *For $z > 0$, the gamma ratio satisfies the double inequality*

$$\left(\frac{z}{z+\alpha}\right)^{1-\alpha} \leq \frac{\Gamma(z+\alpha)}{z^\alpha \Gamma(z)} \leq 1 \qquad (0 \leq \alpha \leq 1).$$

Now we gather some properties of the quantities $\lambda_{n,r}$ and $\mu_{n,r}$.

Lemma 2.2. *The double estimate*

$$1 + \frac{j-1}{2n + j(j-1)} \leq \sqrt[j]{\lambda_{n,j}} \leq 1 + \frac{j-1}{n}$$

is valid for all positive integers n.

Proof. We have

$$\lambda_{n,j} = \frac{n^{\overline{j}}}{n^j} \leq \left(\frac{n+j-1}{n}\right)^j$$

which proves the upper bound. Furthermore, observe that

$$\lambda_{n,j} = \left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{j-1}{n}\right) \geq 1 + \sum_{k=1}^{j-1}\frac{k}{n} = 1 + \frac{1}{n}\binom{j}{2}.$$

Using the inequality

$$\exp\left(\frac{t}{1+t}\right) \leq \sum_{k=0}^{\infty}\left(\frac{t}{1+t}\right)^k = 1+t \qquad (t \geq 0), \tag{5}$$

we obtain

$$\sqrt[j]{\lambda_{n,j}} \geq \left(1 + \frac{1}{n}\binom{j}{2}\right)^{1/j} \geq \exp\left(\frac{1}{j}\frac{\frac{1}{n}\binom{j}{2}}{1+\frac{1}{n}\binom{j}{2}}\right) \geq 1 + \frac{1}{j}\frac{\frac{1}{n}\binom{j}{2}}{1+\frac{1}{n}\binom{j}{2}}$$

and a simplification completes the proof. □

Lemma 2.3. *The double estimate*

$$1 + \frac{j-1}{2n} \leq \sqrt[j]{\mu_{n,j}} \leq 1 + \frac{j-1}{n}$$

is valid for all positive integers $n \geq 2j-1$.

Proof. Because

$$\mu_{n,j}^{-1} = \frac{n^{\underline{j}}}{n^j} = \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{j-1}{n}\right)$$

we obtain

$$\log \mu_{n,j} = -\sum_{k=1}^{j-1} \log\left(1-\frac{k}{n}\right).$$

Taking advantage of the estimate

$$t \le -\log(1-t) \le t + \frac{t^2}{2(1-t)} \qquad (0 \le t < 1)$$

we obtain

$$\sum_{k=1}^{j-1} \frac{k}{n} \le \log \mu_{n,j} \le \sum_{k=1}^{j-1} \frac{k}{n} + \frac{1}{2(1-(j-1)/n)} \sum_{k=1}^{j-1} \left(\frac{k}{n}\right)^2$$

which implies that

$$\frac{j-1}{2n} \le \frac{1}{j} \log \mu_{n,j} \le \frac{j-1}{2n} + \frac{(2j-1)(j-1)}{12n(n+1-j)} = \frac{j-1}{2n}\left(1+\frac{2j-1}{6(n+1-j)}\right).$$

Applying estimate

$$1+t \le e^t \le 1+t+\frac{t^2}{2(1-t)} \qquad (0 \le t < 1)$$

we obtain

$$1+\frac{j-1}{2n} \le \sqrt[j]{\mu_{n,j}} \le 1+t+\frac{t^2}{2(1-t)},$$

where $t = \frac{j-1}{2n}\left(1+\frac{2j-1}{6(n+1-j)}\right)$. If $n \ge 2j-1$, we have $\frac{2j-1}{6(n+1-j)} < 1/3$ and $t \le 1/2$, such that

$$\sqrt[j]{\mu_{n,j}} \le 1+\frac{3}{2}t \le 1+\frac{3}{2}\cdot\frac{j-1}{2n}\cdot\frac{4}{3} \le 1+\frac{j-1}{n}.$$

□

We proceed with some asymptotic relations for $\lambda_{n,r}$ and $\mu_{n,j}$.

Lemma 2.4. *For all integers $r \ge 0$ and $s > 2$, the asymptotic relations*

$$\lambda_{n,r} = 1+\frac{r(r-1)}{2n} + O\left(\frac{1}{n^2}\right), \tag{6}$$

$$\frac{\lambda_{n,rj}}{\lambda_{n,j}^r} = 1 + j^2\frac{r(r-1)}{2n} + O\left(\frac{1}{n^2}\right)$$

and

$$\sum_{r=0}^{s} (-1)^{s-r} \binom{s}{r} \frac{\lambda_{n,rj}}{\lambda_{n,j}^r} = o\left(\frac{1}{n}\right)$$

are valid as $n \to \infty$.

Proof. The first relation follows from

$$\lambda_{n,r} = \left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{r-1}{n}\right) = 1+\frac{1}{n}\sum_{k=1}^{r-1} k + O\left(\frac{1}{n^2}\right) \qquad (n \to \infty).$$

Furthermore, we have

$$\frac{\lambda_{n,rj}}{\lambda_{n,j}^r} = \left(1+\frac{rj(rj-1)}{2n} + O\left(\frac{1}{n^2}\right)\right)\left(1+\frac{j(j-1)}{2n} + O\left(\frac{1}{n^2}\right)\right)^{-r}$$
$$= 1+\frac{rj(rj-1)}{2n} - r\frac{j(j-1)}{2n} + O\left(\frac{1}{n^2}\right) \qquad (n \to \infty).$$

Observe that $rj(rj-1) - rj(j-1) = j^2 r(r-1)$. The last relation is valid since

$$\sum_{r=0}^{s} (-1)^{s-r} \binom{s}{r} r^m = 0,$$

for each nonnegative integer $m < s$. □

Lemma 2.5. *For all integers $r \geq 0$ and $s > 2$, the asymptotic relations*

$$\mu_{n,r} = 1 + \frac{r(r-1)}{2n} + O\left(\frac{1}{n^2}\right), \tag{7}$$

$$\frac{\mu_{n,rj}}{\mu_{n,j}^r} = 1 + j^2 \frac{r(r-1)}{2n} + O\left(\frac{1}{n^2}\right)$$

and

$$\sum_{r=0}^{s} (-1)^{s-r} \binom{s}{r} \frac{\mu_{n,rj}}{\mu_{n,j}^r} = o\left(\frac{1}{n}\right)$$

are valid as $n \to \infty$.

Proof. The first relation follows from

$$\mu_{n,r}^{-1} = \frac{n^{\underline{r}}}{n^r} = \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right) = 1 - \frac{1}{n}\binom{r}{2} + O\left(\frac{1}{n^2}\right) \qquad (n \to \infty).$$

Now, the further relations follow as in the proof of Lemma 2.4. □

3 Related operators preserving e_0 and e_j

In this section we introduce two types of modifications of the operators P_n and G_n. Let $j \geq 1$ be an integer number. The new operators $P_{n,j}$, $G_{n,j}$, $\tilde{P}_{n,j}$ and $\tilde{G}_{n,j}$ preserve the monomials e_0 and e_j.

The first pair of modified operators is defined as follows

$$\left(P_{n,j} f(t)\right)(x) := \left(P_n f\left(\frac{t}{\sqrt[j]{\lambda_{n,j}}}\right)\right)(x)$$

and

$$\left(G_{n,j} f(t)\right)(x) := \left(G_n f\left(\frac{t}{\sqrt[j]{\mu_{n,j}}}\right)\right)(x).$$

Then $P_{n,j} e_0 = e_0$, $G_{n,j} e_0 = e_0$, and

$$P_{n,j} e_j = P_n \frac{e_j}{\lambda_{n,j}} = e_j,$$

$$G_{n,j} e_j = G_n \frac{e_j}{\mu_{n,j}} = e_j.$$

In the following we study the moments of the operators $P_{n,j}$ and $G_{n,j}$. By definition and equations (2) and (3), we have

$$P_{n,j} e_r = \frac{\lambda_{n,r}}{\lambda_{n,j}^{r/j}} e_r \qquad \text{and} \qquad G_{n,j} e_r = \frac{\mu_{n,r}}{\mu_{n,j}^{r/j}} e_r,$$

where $r \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$. As a consequence of (6) and (7), we get the following proposition.

Proposition 3.1. *For $r \in \mathbb{N}_0$, the moments of the operators $P_{n,j}$ and $G_{n,j}$ satisfy the pointwise asymptotic relations*

$$P_{n,j} e_r = \left[1 + \frac{r(r-j)}{2n} + O\left(\frac{1}{n^2}\right)\right] e_r \qquad (n \to \infty),$$

$$G_{n,j} e_r = \left[1 + \frac{r(r-j)}{2n} + O\left(\frac{1}{n^2}\right)\right] e_r \qquad (n \to \infty).$$

The second type of modification is given by

$$\left(\tilde{P}_{n,j} f(t)\right)(x) = \left(P_n f\left(\sqrt[j]{t}\right)\right)(x^j) = \frac{j n^n}{x^{nj}(n-1)!} \int_0^\infty u^{nj-1} e^{-nu^j/x^j} f(u)du$$

and

$$\left(\tilde{G}_{n,j} f(t)\right)(x) = \left(G_n f\left(\sqrt[j]{t}\right)\right)(x^j) = \frac{(nx^j)^{n+1}}{n!} j \int_0^\infty \frac{1}{u^{(n+1)j+1}} e^{-nx^j/u^j} f(u)du.$$

By definition and (1.1), we have

$$\tilde{P}_{n,j} e_r = \frac{\Gamma(n + r/j)}{n^{r/j}\Gamma(n)} e_r \qquad \text{and} \qquad \tilde{G}_{n,j} e_r = \frac{\Gamma(n+1-r/j)}{n^{-r/j}\Gamma(n+1)} e_r. \tag{8}$$

These equations reveal that $\tilde{P}_{n,j}$ and $\tilde{G}_{n,j}$ preserve the monomials e_0 and e_j.

The asymptotic behaviour of the moments is described in the following proposition.

Proposition 3.2. *For $r \in \mathbb{N}_0$, the moments of the operators $\tilde{P}_{n,j}$ and $\tilde{G}_{n,j}$ satisfy the pointwise asymptotic relations*

$$\tilde{P}_{n,j}e_r = \left[1 + \frac{r(r-j)}{2j^2 n} + O\left(n^{-2}\right)\right]e_r, \text{ as } n \to \infty, \tag{9}$$

$$\tilde{G}_{n,j}e_r = \left[1 + \frac{r(r-j)}{2j^2 n} + O\left(n^{-2}\right)\right]e_r, \text{ as } n \to \infty. \tag{10}$$

Proof. Let $r \in \mathbb{N}_0$. By (8) and (4) we have

$$\tilde{P}_{n,j}e_r = \left[1 + \frac{r/j\,(r/j-1)}{2n} + O\left(n^{-2}\right)\right]e_r \text{ as } n \to \infty,$$

which proves the first relation. Analogously, we have

$$\tilde{G}_{n,j}e_r = \frac{n-r/j}{n}\frac{\Gamma(n-r/j)}{n^{-r/j}\Gamma(n)}e_r = \left(1 - \frac{r}{jn}\right)\left[1 + \frac{-r/j\,(-r/j-1)}{2n} + O\left(n^{-2}\right)\right]e_r$$

as $n \to \infty$ and the second relation follows by simplification. □

4 Convergence of the operators

Let $C_b[0,+\infty)$ be the space of all real-valued, continuous and bounded functions defined on the interval $[0,+\infty)$. In this section we use the notation $\psi_x(t) = t - x$.

In order to show the convergence we can apply the classical estimate using the second central moment (see [5, Theorem 5.1.2]).

Lemma 4.1. *If $Le_0 = e_0$, then, for $f \in C_b[0,+\infty)$,*

$$|(Lf)(x) - f(x)| \le \left(1 + \frac{1}{\delta}\sqrt{\left(L\psi_x^2\right)(x)}\right)\omega(f,\delta).$$

4.1 Convergence of the operators $P_{n,j}$

First we consider the operators $P_{n,j}$ and prove the convergence (see Theorem 4.3).

Proposition 4.2. *For $x > 0$ and positive integers n, the second central moment of $P_{n,j}$ satisfies the estimate*

$$\left(P_{n,j}\psi_x^2\right)(x) \le \left(\frac{1}{n} + \frac{(j-1)^2}{n^2}\right)x^2. \tag{11}$$

If $n \ge j(j-1)$, it follows the more elegant estimate

$$\left(P_{n,j}\psi_x^2\right)(x) \le \frac{2j-1}{jn}x^2. \tag{12}$$

Proof. We have

$$\begin{aligned}\left(P_{n,j}\psi_x^2\right)(x) &= \lambda_{n,j}^{-2/j}(P_n e_2)(x) - 2x\lambda_{n,j}^{-1/j}(P_n e_1)(x) + x^2 P_n(e_0)(x) \\ &= x^2\left[\frac{1}{n\lambda_{n,j}^{2/j}} + \left(1 - \frac{1}{\lambda_{n,j}^{1/j}}\right)^2\right].\end{aligned}$$

By Lemma 2.2,

$$1 \le \sqrt[j]{\lambda_{n,j}} \le 1 + \frac{j-1}{n}.$$

Therefore, we have

$$\frac{1}{n\lambda_{n,j}^{2/j}} + \left(1 - \frac{1}{\lambda_{n,j}^{1/j}}\right)^2 \le \frac{1}{n} + \left(\frac{\frac{j-1}{n}}{1 + \frac{j-1}{n}}\right)^2 = \frac{1}{n} + \left(\frac{j-1}{n+j-1}\right)^2$$

and so,

$$\left(P_{n,j}\psi_x^2\right)(x) \le \left(\frac{1}{n} + \frac{(j-1)^2}{n^2}\right)x^2,$$

which is (11).

For $n \ge j(j-1)$ we have

$$\frac{1}{n} + \frac{(j-1)^2}{n^2} \le \frac{2j-1}{jn}.$$

Thus (12) holds true.

□

Remark 1. Note that $\left(P_{n,j}\psi_x^2\right)(x) = \frac{x^2}{n} + O\left(n^{-2}\right)$ as $n \to \infty$.

Theorem 4.3. *Let $x > 0$. For $f \in C_b[0,+\infty)$, the rate of convergence can be estimated by*

$$\left|\left(P_{n,j}f\right)(x) - f(x)\right| \le \left(1 + \sqrt{\frac{2j-1}{j}x}\right)\omega\left(f, \frac{1}{\sqrt{n}}\right), \tag{13}$$

for all integers $n \ge j(j-1)$.

Proof. By Lemma 4.1 and Proposition 4.2, for $f \in C_b[0,+\infty)$ and $\delta > 0$,

$$\left|\left(P_{n,j}f\right)(x) - f(x)\right| \le \left(1 + \frac{x}{\delta}\sqrt{\frac{2j-1}{jn}}\right)\omega(f,\delta).$$

Choosing $\delta = 1/\sqrt{n}$ we get (13). □

4.2 Convergence of the operators $G_{n,j}$

Theorem 4.4. *Let $j \ge 2$ and $x > 0$. For $f \in C_b[0,+\infty)$, the rate of convergence can be estimated by*

$$\left|\left(G_{n,j}f\right)(x) - f(x)\right| \le \left(1 + \sqrt{j}x\right)\omega\left(f, \frac{1}{\sqrt{n}}\right),$$

for all integers $n \ge j-1$.

Proof. For $j \ge 2$ and $x > 0$, by the definition and (1), we obtain

$$\begin{aligned}\left(G_{n,j}\psi_x^2\right)(x) &= \mu_{n,j}^{-2/j}(G_n e_2)(x) - 2x\mu_{n,j}^{-1/j}(G_n e_1)(x) + x^2(G_n e_0)(x)\\ &= x^2\left[\frac{1}{(n-1)\mu_{n,j}^{2/j}} + \left(1 - \frac{1}{\mu_{n,j}^{1/j}}\right)^2\right].\end{aligned}$$

Because

$$\mu_{n,j}^{-1} = \frac{n^{\underline{j}}}{n^j} = \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{j-1}{n}\right) \le \left(1-\frac{1}{n}\right)^{j-1},$$

we infer that $\mu_{n,j}^{-2/j} \le \left(1-\frac{1}{n}\right)^{2-2/j} \le 1 - \frac{1}{n}$, such that

$$\frac{1}{(n-1)\mu_{n,j}^{2/j}} \le \frac{1}{n}.$$

Furthermore, we have the estimate

$$\mu_{n,j}^{-1/j} \ge \left(1 - \frac{j-1}{n}\right)^{(j-1)/j} \ge 1 - \frac{j-1}{n} \qquad (n \ge j-1).$$

Therefore, we obtain,

$$\left(G_{n,j}\psi_x^2\right)(x) \le x^2\left[\frac{1}{n} + \left(\frac{j-1}{n}\right)^2\right] \le \frac{j}{n}x^2 \qquad (n \ge j-1).$$

and this concludes the proof. □

4.3 Convergence of the operators $\tilde{P}_{n,j}$

Theorem 4.5. *Let $j \ge 2$ and $x > 0$. For $f \in C_b[0,+\infty)$, the rate of convergence can be estimated by*

$$\left|\left(\tilde{P}_{n,j}f\right)(x) - f(x)\right| \le \left(1 + \sqrt{2\frac{j-1}{j}x}\right)\omega\left(f, \frac{1}{\sqrt{n}}\right), \tag{14}$$

for all integers $n \ge 1$.

Proof. For $j \ge 2$, by the definition and (1), we obtain

$$\begin{aligned}\left(\tilde{P}_{n,j}\psi_x^2\right)(x) &= P_n\left(t^{2/j}; x^j\right) - 2xP_n\left(t^{1/j}; x^j\right) + x^2 P_n\left(1; x^j\right)\\ &= x^2\frac{\Gamma(n+2/j)}{n^{2/j}\Gamma(n)} - 2x^2\frac{\Gamma(n+1/j)}{n^{1/j}\Gamma(n)} + x^2.\end{aligned}$$

Applying Wendel's inequality (Lemma 2.1) in the form

$$(n+\alpha-1)^{\alpha} \leq \frac{\Gamma(n+\alpha)}{\Gamma(n)} \leq n^{\alpha} \qquad (n > 1-\alpha, \quad 0 \leq \alpha \leq 1),$$

(note that $0 < 2/j \leq 1$) we infer that

$$\begin{aligned}\left(\tilde{P}_{n,j}\psi_x^2\right)(x) &\leq x^2\left(1-2\frac{(n+1/j-1)^{1/j}}{n^{1/j}}+1\right)\\ &= 2x^2\left(1-\left(1-\frac{j-1}{jn}\right)^{1/j}\right).\end{aligned}$$

Using the inequality $\sqrt[j]{1-t} \geq 1-t$, for $t \in [0,1]$ and $j \in \mathbb{N}$, we conclude that

$$\left(\tilde{P}_{n,j}\psi_x^2\right)(x) \leq 2x^2\frac{j-1}{jn} \qquad (n \in \mathbb{N}).$$

Then, for $f \in C_b[0,+\infty)$ and $\delta > 0$,

$$\left|\left(\tilde{P}_{n,j}f\right)(x) - f(x)\right| \leq \left(1+\frac{x}{\delta}\sqrt{\frac{2}{n}\frac{j-1}{j}}\right)\omega(f,\delta).$$

Choosing $\delta = 1/\sqrt{n}$ we obtain (14). □

4.4 Convergence of the operators $\tilde{G}_{n,j}$

Theorem 4.6. *Let $j \geq 2$ and $x > 0$. For $f \in C_b[0,+\infty)$, the rate of convergence can be estimated by*

$$\left|\left(\tilde{G}_{n,j}f\right)(x) - f(x)\right| \leq \left(1+\sqrt{\frac{2(j-1)}{j}x}\right)\omega\left(f,\frac{1}{\sqrt{n}}\right),$$

for all integers $n \geq 1$.

Proof. By the definition and (1), we have

$$\begin{aligned}\left(\tilde{G}_{n,j}\psi_x^2\right)(x) &= G_n\left(t^{2/j};x^j\right) - 2xG_n\left(t^{1/j};x^j\right) + x^2G_n\left(1;x^j\right)\\ &= x^2\left(\frac{n^{2/j}\Gamma(n+1-2/j)}{\Gamma(n+1)} - 2\frac{n^{1/j}\Gamma(n+1-1/j)}{\Gamma(n+1)}+1\right).\end{aligned}$$

Applying Wendel's inequality (Lemma 2.1) in the equivalent form

$$\left(\frac{x}{x+1-s}\right)^s \leq \frac{x^s\Gamma(x+1-s)}{\Gamma(x+1)} \leq 1 \qquad (x > 0, \quad 0 \leq s \leq 1)$$

we obtain

$$\left(\tilde{G}_{n,j}\psi_x^2\right)(x) \leq 2x^2\left[1-\left(\frac{n}{n+1-1/j}\right)^{1/j}\right] = 2x^2\left[1-\left(1-\frac{(j-1)/j}{n+(j-1)/j}\right)^{1/j}\right]$$

Using the inequality $\sqrt[j]{1-t} \geq 1-t$, for $t \in [0,1]$ and $j \in \mathbb{N}$, we conclude that

$$\left(\tilde{G}_{n,j}\psi_x^2\right)(x) \leq 2x^2\frac{(j-1)/j}{n+(j-1)/j} \leq 2x^2\frac{j-1}{nj} \qquad (n \in \mathbb{N}),$$

and this completes the proof.

□

5 Voronovskaja type results

In order to obtain Voronovskaja type results for the operators $P_{n,j}$, $\tilde{P}_{n,j}$, $G_{n,j}$, $\tilde{G}_{n,j}$ we use the following general result (see [2, Section 2]).

Let $\varphi \in C^2[0,\infty)$, $\varphi(0)=0$, $\varphi'(t)>0$, $t \in (0,\infty)$, $\lim_{t\to\infty}\varphi(t)=\infty$. Denote

$$E_{\varphi} := \left\{f \in C[0,\infty) \,\middle|\, \sup_{t\geq 0}\frac{|f(t)|}{1+\varphi^2(t)} < \infty\right\}.$$

Theorem 5.1. *[2] Let $x > 0$ be given and let $\Psi_x(t) := \varphi(t) - \varphi(x)$, $t \geq 0$. Denote by E_{φ}^x a linear subspace of $C[0,\infty)$ such that $E_{\varphi} \subset E_{\varphi}^x$ and $\Psi_x^4 \in E_{\varphi}^x$. Let $L_n : E_{\varphi}^x \to C[0,\infty)$ be a sequence of positive linear operators such that*

(i) $\lim_{n\to\infty} n((L_ne_0)(x)-1) = 0,$

(ii) $\lim_{n\to\infty} n(L_n\Psi_x)(x) = b(x)$,

(iii) $\lim_{n\to\infty} n(L_n\Psi_x^2)(x) = 2a(x)$,

(iv) $\lim_{n\to\infty} n(L_n\Psi_x^4)(x) = 0$.

If $f \in E_\varphi$ and there exists $f''(x) \in \mathbb{R}$, then

$$\lim_{n\to\infty} n((L_n f)(x) - f(x)) = \frac{a(x)}{\varphi'(x)^2} f''(x) + \frac{b(x)\varphi'(x)^2 - a(x)\varphi''(x)}{\varphi'(x)^3} f'(x). \tag{15}$$

Let us mention that in [2] the important idea in connection with the application of Theorem 5.1 to modified Baskakov type operators preserving the constants and x^j was to use $\varphi(t) = t^j$. In similar investigations usually $\varphi(t) = t$ (classical case) or $\varphi(t) = t^2$ was used (see [6]), which didn't lead to a possible proof for Voronovskaja type result for the modified Baskakov operator.

We now apply Theorem 5.1 to the operators $\tilde{P}_{n,j}$, $\tilde{G}_{n,j}$ with $\varphi(t) = t^j$.

Theorem 5.2. *Let $x > 0$, $f \in C[0,\infty)$ with $\sup_{t\geq 0} \frac{|f(t)|}{1+t^{2j}} < \infty$, such that there exists $f''(x) \in \mathbb{R}$. The sequences $(\tilde{P}_{n,j})_{n\geq 1}$ and $(\tilde{G}_{n,j})_{n\geq 1}$ satisfy the Voronovskaja type formulas*

$$\lim_{n\to\infty} n\left((\tilde{P}_{n,j} f)(x) - f(x)\right) = \frac{1}{j^2}\left(\frac{x^2}{2} f''(x) - \frac{(j-1)x}{2} f'(x)\right),$$
$$\lim_{n\to\infty} n\left((\tilde{G}_{n,j} f)(x) - f(x)\right) = \frac{1}{j^2}\left(\frac{x^2}{2} f''(x) - \frac{(j-1)x}{2} f'(x)\right).$$

Proof. As $\tilde{P}_{n,j}$ and $\tilde{G}_{n,j}$ preserve e_0 and e_j, (i) and (ii) in Theorem 5.1 are fulfilled with $b(x) = 0$.

Taking $r = lj$, $l \in \mathbb{N}$, in (8), we get

$$(\tilde{P}_{n,j} e_{lj})(x) = x^{lj} \frac{n^{\bar{l}}}{n^l}$$

and

$$(\tilde{P}_{n,j}\Psi_x^2)(x) = x^{2j}\frac{1}{n},\ (\tilde{P}_{n,j}\Psi_x^4)(x) = x^{4j}\frac{3(n+2)}{n^3}.$$

Thus (iii) with $2a(x) = x^{2j}$ and (iv) in Theorem 5.1 are also valid, which leads to the proposed Voronovskaja type result for $\tilde{P}_{n,j}$.

Taking $r = lj$, $l \in \mathbb{N}$, in (8), we get

$$(\tilde{G}_{n,j} e_{lj})(x) = x^{lj} \frac{n^l}{n^{\underline{l}}}$$

and

$$(\tilde{G}_{n,j}\Psi_x^2)(x) = x^{2j}\frac{1}{n-1},\ (\tilde{G}_{n,j}\Psi_x^4)(x) = x^{4j}\frac{3(n+6)}{(n-1)(n-2)(n-3)},$$

which leads to the desired Voronovskaja type result for $\tilde{G}_{n,j}$.

□

Remark 2. Concerning the proof of Theorem 5.2, let us remark in passing that conditions (i)-(iv) from Theorem 5.1 for the operators $\tilde{P}_{n,j}$ and $\tilde{G}_{n,j}$ also follow easily from (9) and (10).

In the following we investigate Voronovskaja type formulas for the operators $P_{n,j}$ and $G_{n,j}$. Clearly, we have $\left(P_{n,j}\Psi_x^0\right)(x) = \left(G_{n,j}\Psi_x^0\right)(x) = 1$ and $\left(P_{n,j}\Psi_x^1\right)(x) = \left(G_{n,j}\Psi_x^1\right)(x) = 0$. Moreover, we show the following formulas.

Theorem 5.3. *Let $x > 0$, $f \in C[0,\infty)$ with $\sup_{t\geq 0} \frac{|f(t)|}{1+t^{2j}} < \infty$, such that there exists $f''(x) \in \mathbb{R}$. The sequences $(P_{n,j})_{n\geq 1}$ and $(G_{n,j})_{n\geq 1}$ satisfy the Voronovskaja type formulas*

$$\lim_{n\to\infty} n\left((P_{n,j} f)(x) - f(x)\right) = \frac{x^2}{2} f''(x) - \frac{(j-1)x}{2} f'(x),$$
$$\lim_{n\to\infty} n\left((G_{n,j} f)(x) - f(x)\right) = \frac{x^2}{2} f''(x) - \frac{(j-1)x}{2} f'(x).$$

Proof. We have

$$\left(P_{n,j}\Psi_x^s\right)(x) = \sum_{r=0}^{s}\binom{s}{r}\left(-x^j\right)^{s-r}\left(P_{n,j}e_{rj}\right)(x) = x^{js}\sum_{r=0}^{s}(-1)^{s-r}\binom{s}{r}\frac{\lambda_{n,rj}}{\lambda_{n,j}^r}.$$

Therefore, by Lemma 2.4,

$$\left(P_{n,j}\Psi_x^2\right)(x) = x^{2j}\left(\frac{\lambda_{n,2j}}{\lambda_{n,j}^2} - 2\frac{\lambda_{n,j}}{\lambda_{n,j}} + 1\right) = \frac{j^2}{n}x^{2j} + O\left(\frac{1}{n^2}\right)$$

and $\left(P_{n,j}\Psi_x^4\right)(x) = o\left(n^{-1}\right)$ as $n \to \infty$. The same formulas are valid for the operator $G_{n,j}$.

So, we get $b(x) = 0$ and $2a(x) = j^2 x^{2j}$. Using Theorem 5.1 with $\varphi(t) = t^j$, concludes the proof. □

Remark 3. A more elaborate analysis shows that

$$\begin{aligned}
\left(P_{n,j}\Psi_x^2\right)(x) &= \frac{j^2}{n}x^{2j} + \frac{j^2(j-1)^2}{2n^2}x^{2j} + O\left(\frac{1}{n^3}\right),\\
\left(P_{n,j}\Psi_x^3\right)(x) &= \frac{j^3(3j-1)}{n^2}x^{3j} + O\left(\frac{1}{n^3}\right),\\
\left(P_{n,j}\Psi_x^4\right)(x) &= \frac{3j^4}{n^2}x^{4j} + O\left(\frac{1}{n^3}\right),\\
\left(G_{n,j}\Psi_x^2\right)(x) &= \frac{j^2}{n}x^{2j} + \frac{j^2\left(j^2+2j-1\right)}{2n^2}x^{2j} + O\left(\frac{1}{n^3}\right),\\
\left(G_{n,j}\Psi_x^3\right)(x) &= \frac{j^3(3j+1)}{n^2}x^{3j} + O\left(\frac{1}{n^3}\right),\\
\left(G_{n,j}\Psi_x^4\right)(x) &= \frac{3j^4}{n^2}x^{4j} + O\left(\frac{1}{n^3}\right)
\end{aligned}$$

as $n \to \infty$.

Remark 4. The efficiency of the approximation furnished by a sequence of operators can be measured, in particular, by the magnitude of its Voronovskaja limit. Comparing the results for the different modifications it is worth mentioning that in general for fixed $j \geq 2$ the absolute value of the Voronovskaja limit for $\tilde{P}_{n,j}$ and $\tilde{G}_{n,j}$ is smaller than for $P_{n,j}$ and $G_{n,j}$ by a factor of j^{-2}.

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